A Port-Hamiltonian Approach to Distributed Parameter Systems

Dutch Institute of Systems and Control

The research described in this thesis was undertaken at the Department of Applied Mathematics, in the Faculty EWI, Universiteit Twente, Enschede. The funding of the research was provided by the NWO Grant through project number 613.000.223 (Energy-based representation, analysis, and control of infinite-dimensional systems, ERACIS).

Printed by Wöhrmann Printing Service, Zutphen, The Netherlands.

The summary in Dutch was done by H.J. Zwart and A.J. van der Schaft.

ISBN: 978-90-365-2489-6

A PORT-HAMILTONIAN APPROACH TO DISTRIBUTED PARAMETER SYSTEMS

DISSERTATION

to obtain
the doctor's degree at the University of Twente,
on the authority of the rector magnificus,
prof. dr. W.H.M. Zijm,
on account of the decision of the graduation committee,
to be publicly defended
on Friday 11 May 2007 at 13.15 hours

by

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This dissertation has been approved by the promotor
Prof. dr. A. J. van der Schaft
and the assistant promotor
dr. H. J. Zwart

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Summary

This thesis aims to provide a mathematical framework for the modeling and analysis of open distributed parameter systems. From a mathematical point of view this thesis merges the approach based on Hamiltonian modeling of open distributed-parameter systems, employing the notion of port-Hamiltonian systems, with the semigroup approach of infinite-dimensional systems theory. The Hamiltonian representation provides powerful analysis methods (e.g. for stability), and it enables the use of Lyapunov-stability theory and passivity-based control. The semigroup approach has been widely applied in the analysis of distributed parameter systems and it has facilitated the extension of some notions from finite-dimensional system theory to the infinite-dimensional case.

One of the key points of the port-Hamiltonian formulation is the structure of the mathematical model obtained. By exploiting this structure, the port-Hamiltonian approach allows to deal with classes of systems, which provide a relative new point of view in the analysis of distributed parameter systems. In this thesis the port-Hamiltonian formulation is mainly used for the analysis of 1D-boundary control systems. These are systems in which the input (or part of it) acts on the boundary of the spatial domain. In these cases it is possible to parameterize the selection of the inputs (boundary conditions) and outputs by the selection of two matrices in such a way that the resulting system is passive. In this case these matrices determine the supply rate of the passive system, making it easy, in particular, to obtain impedance passive and scattering passive systems. In fact, as it is shown, these matrices can be used to determine further properties of the system, such as stability, controllability, and observability. Furthermore, it is shown that this approach already covers a very large class of 1D-systems. This thesis treats mainly two broad classes of systems. One corresponds to systems where the dissipation phenomena is not present and the other includes systems with some type of dissipation (e.g. heat or mass transfer, damping). These classes can, in turn, be divided into subclasses according to the properties of the structure, to provide further tools for the analysis of such systems.

Thus the structure of the resulting models forms the basis for the development of general analysis (and control) techniques. In fact, it is shown that for some classes of systems it is possible to easily determine some of their fundamental properties (e.g. existence of solutions, stability, Riesz basis property). In this thesis we provide simple tools for the analysis of these properties for some classes of systems.

Samenvatting

Dit proefschrift probeert een wiskundige kader te geven voor de modellering en de analyse van open verdeelde-parameter systemen. Vanuit een wiskundig standpunt verbindt dit proefschrift de Hamiltoniaanse benadering van open verdeelde-parameter systemen, gebruik makend van poort-Hamiltoniaanse systemen, met de half-groep benadering uit de oneindig-dimensionale systeemtheorie. De Hamiltoniaanse beschrijving geeft krachtige methodes voor de analyse, bijvoorbeeld met betrekking tot de stabiliteit. Verder maakt het een Lyapunov stabiliteitstheorie en een regelontwerp gebaseerd op passiviteit mogelijk. De half-groep benadering is veel toegepast binnen de analyse van oneindig-dimensionale systemen, en begrippen uit de eindig-dimensionale systeemtheorie zijn uitgebreid naar deze klasse.

Eén van de kararakteristieken van poort-Hamiltoniaanse systemen is de structuur in het wiskundige model. Deze structuur maakt het mogelijk om een klasse van systemen te beschouwen, en het geeft een nieuwe benadering voor de analyse van verdeelde-parameter systemen. In het proefschrift wordt de poort-Hamiltoniaanse benadering voornamelijk gebruikt voor de analyse van 1-D systemen met randbesturing. Dit zijn systemen waar de besturing werkt op de rand van een eendimensionaal plaatselijk domein. Voor deze klasse is het mogelijk om door middel van de keuze van twee matrices de in- en uitgangen te selecteren opdat het systeem passief is. De matrices bepalen de expressie van het toegeleverd vermogen van het systeem, en bepalen daarmee in welke zin het systeem passief is. Verder tonen we aan dat deze matrices gebruikt kunnen worden om andere systeemeigenschappen, zoals stabiliteit, regelbaarheid, en waarneembaarheid, te bewijzen. Deze aanpak is toepasbaar op een zeer grote klasse van 1-D systemen. Dit proefschrift behandelt twee ruime klassen van systemen. De eerste klasse zijn systemen zonder interne dissipatie, en in de tweede klasse is deze dissipatie wel aanwezig, bijvoorbeeld door warmte- of massatransport, of door demping. Deze klassen kunnen op grond van hun structuureigenschappen verder opgedeeld worden. Dit geeft extra gereedschappen voor de analyse van deze systemen.

Dus de structuur van de modellen vormt de basis voor de ontwikkeling van een algemene techniek voor zowel de analyse als voor regelaarontwerp. In het bijzonder wordt aangetoond dat voor een deelklasse van systemen het mogelijk is om op eenvoudige wijze fundamentele eigenschappen te bewijzen. Dit geldt onder andere voor eigenschappen zoals het bestaan van oplossingen, en het bezit-

ten van een Riesz basis van eigenvectoren. In dit proefschrift ontwikkelen we eenvoudige gereedschappen voor de analyse van onze klasse van systemen.

Notation

Symbol	Description	Page
$\mathbb{R}, \mathbb{R}_+$	real numbers, positive real numbers	13
$L_2(a,b;\mathbb{R}^n)$ (or $L_2(a,b)^n$)	vector space of square integrable functions on $\mathbb{R}^n$	13
$H^N(a,b;\mathbb{R}^n)$ (or $H^N(a,b)^n$)	Sobolev space of order N	13
$\mathcal{D}(\Omega)$	space of all indefinitely differentiable functions with a compact support in Ω	13
$\mathcal{D}'(\Omega)$	dual space of $\mathcal{D}(\Omega)$	13
$\mathcal{D}(\overline{\Omega})$	space of all indefinitely differentiable functions	199
$\mathcal{L}(X,Y)$	space of bounded linear operators from X to Y	13
$\mathcal{L}(X)$	space of bounded linear operators on X	13
$M_{n\times m}(Y)$	set of $n \times m$ matrices with entries in the space Y	13
$M_n(Y)$	set of $n \times n$ matrices with entries in the space Y	13
$\rho(T)$	resolvent set of T	13
$\sigma(T)$	spectrum of T	13
$\sigma_p(T)$	point spectrum of T	88

Symbol	Description	Page
$T_{\|H}$	restriction of T to the space H.	13
I_X (or I)	identity operator on X	14
$\\|\cdot\\|$	norm on $L_2(a,b)$ (or $L_2(a,b)^n$)	13
$\\|\cdot\\|_H$	norm on H (or H^n)	13
$\langle \cdot, \cdot \rangle$	inner product on $L_2(a,b)$ (or $L_2(a,b)^n$)	6
$\langle \cdot, \cdot \rangle_H$	inner product on H (or H^n)	13
$\langle\langle \cdot, \cdot \rangle\rangle$	duality product on $\mathcal{D}(\Omega) \times \mathcal{D}'(\Omega)$	199
$\langle\langle \cdot, \cdot \rangle\rangle_H$	duality product on $H \times H'$	13
$\frac{dx}{ds}$	derivative of $x(s)$ with respect to s	14
$\frac{dx}{dt}$ (or $\dot{x}$)	derivative of $x(t)$ with respect to time t	14
$\frac{\partial w}{\partial x_1}$ (or $\partial_{x_1} w$)	partial derivative of $w(x_1, \ldots, x_n)$ with respect to x_1	14
$\frac{\delta H}{\delta x}$	variational derivative of the functional $H(x)$	14
∇	gradient operator,	206
div $(\cdot)$	divergence operator	206
$\rho * x$	convolution product between ρ and x	209

Chapter 1

Introduction

The first part of this chapter provides a summary and background information about the results. In the second part we present the structure of this book, and highlight the contributions made in it.

In this chapter the reader might find a few terminologies that have not been explained or some ideas that appear vague. They will be explained later in the following chapters. They are included in this chapter for the ease and completeness of presentation. For ease of reference there is included in this book a notation table and an index where some terms and definitions can be found.

1.1. Motivation

In order to motivate and show the relevance of the theory presented in this book, we give some simple examples of control problems that arise for distributed parameter systems, in particular boundary control systems.

Example 1.1 (Wave equation) Consider a *vibrating string* of length $L = b - a$, held stationary at both ends and free to vibrate transversely subject to the restoring forces due to tension in the string. The vibrations on the system can be modeled by

$$\frac{\partial^2 u}{\partial t^2}(z,t) = c\frac{\partial^2 u}{\partial z^2}(z,t), \qquad c = \frac{T}{\rho},\ t \geq 0, \tag{1.1}$$

where $z \in [a, b]$ is the spatial variable, $u(z, t)$ is the vertical position of the string, $T(z)$ is the Young's modulus of the string, and $\rho(z)$ is the mass density. This model is a simplified version of other systems where vibrations occur, as in the case of large structures, and it is also used in acoustics. In this case, the control problem is to damp out the vibrations on the string. One approach to do this is to add damping along the spatial domain. This can also be done by interacting with the forces and velocities at the end of the string, i.e., at the boundary. $*$

Example 1.2 (Beam equations) In recent years the boundary control of flexible structures has attracted much attention with the increase of high technology applications such as space science and robotics. In these applications the control of vibrations is crucial. These vibrations can be modeled by beam equations. For instance, the *Euler-Bernoulli beam* equation models the transversal vibration of an elastic beam if the cross-section dimension of the beam is negligible in comparison with its length. If the cross-section dimension is not negligible, then it is necessary to consider the effect of the rotary inertia. In that case, the transversal vibration is better described by the *Rayleigh beam equation*. An improvement over these models is given by the *Timoshenko beam*, since it incorporates shear and rotational inertia effects, which makes it a more precise model. These equations are given, respectively, by

- Euler-Bernoulli beam:

$$\rho(z)\frac{\partial^2 w}{\partial t^2}(z,t) + \frac{\partial^2}{\partial z^2}\left(EI(z)\frac{\partial^2 w}{\partial z^2}(z,t)\right) = 0, \quad z \in (a,b),\ t \geq 0,$$

  where $w(t,z)$ is the transverse displacement of the beam, $\rho(z)$ is the mass per unit length, $E(z)$ is the Young's modulus of the beam, and $I(z)$ is the area moment of inertia of the beam's cross section.

- Rayleigh beam:

$$\rho(z)\frac{\partial^2 w}{\partial t^2}(z,t) - I_\rho(z)\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial z^2}(z,t)\right) + \frac{\partial^2}{\partial z^2}\left(EI(z)\frac{\partial^2 w}{\partial z^2}(z,t)\right) = 0,$$

  where $z \in (a,b)$, $t \geq 0$, $w(t,z)$ is the transverse displacement of the beam, $\rho(z)$ is the mass per unit length, I_ρ is the rotary moment of inertia of a cross section, $E(z)$ is the Young's modulus of the beam, and $I(z)$ is the area moment of inertia.

- Timoshenko beam:

$$\begin{aligned} \rho(z)\frac{\partial^2 w}{\partial t^2}(z,t) &= \frac{\partial}{\partial z}\left[K(z)\left(\frac{\partial w}{\partial z}(z,t) - \phi(z,t)\right)\right], \quad z \in (a,b),\ t \geq 0, \\ I_\rho(z)\frac{\partial^2 \phi}{\partial t^2}(z,t) &= \frac{\partial}{\partial z}\left(EI(z)\frac{\partial \phi}{\partial z}(z,t)\right) + K(z)\left(\frac{\partial w}{\partial z}(z,t) - \phi(z,t)\right), \end{aligned}$$

  where $w(t,z)$ is the transverse displacement of the beam and $\phi(t,z)$ is the rotation angle of a filament of the beam. The coefficients $\rho(z)$, $I_\rho(z)$, $E(z)$, $I(z)$, and $K(z)$ are the mass per unit length, the rotary moment of inertia of a cross section, Youngs modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively. $*$

Example 1.3 (Suspension system) Consider a simplified version of a suspension system described by two strings connected in parallel through a distributed spring. This system can be modeled by

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial z^2} + \alpha(v-u) \\ \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial z^2} + \alpha(u-v) \end{aligned} \qquad z \in (a,\, b),\ t \geq 0, \tag{1.2}$$

where c and α are positive constants and $u(z,t)$ and $v(z,t)$ describe the displacement, respectively, of both strings. The use of this model has potential applications in isolation of objects from outside disturbances. As an example in engineering, rubber and rubber-like materials are used to absorb vibration or shield structures from vibration. As an approximation, these materials can be modeled as a distributed spring. Modeling of structures such as beams, or plates sandwiched with rubber or similar materials, will lead to equations similar to those in (1.2). Later we show that this system can be described as the interconnection of three subsystems, i.e., two vibrating strings and one distributed spring. Seeing the system as an interconnection of subsystems allows to have some modularity in the modeling process, and because of this modularity, the modeling process can be performed in an iterative manner, gradually refining the model by adding other subsystems. $*$

Example 1.4 (Heat conduction) The model of heat conduction consists of only one conservation law, that is the *conservation of energy*. It is given by the following conservation law:

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial z} J_Q, \tag{1.3}$$

where $u(z,t)$ is the energy density and $J_Q(z,t)$ is the heat flux. This conservation law is completed by two closure equations. The first one expresses the calorimetric properties of the material :

$$\frac{\partial u}{\partial T} = c_V(T), \tag{1.4}$$

where $T(z,t)$ is the temperature distribution and c_V is the heat capacity. The second closure equation defines heat conduction property of the material (Fourier's conduction law):

$$J_Q = -\lambda(T,z)\frac{\partial T}{\partial z}, \tag{1.5}$$

where $\lambda(T,z)$ denotes the heat conduction coefficient. Assuming that the variations of the temperature are not too large, one may assume that the heat capacity and the heat conduction coefficient are independent of the temperature, one obtains the following partial differential equation:

$$\frac{\partial T}{\partial t} = \frac{1}{c_V}\frac{\partial}{\partial z}\left(\lambda(z)\frac{\partial T}{\partial z}\right). \tag{1.6}$$

$*$

Example 1.5 (The fixed bed reactor) Another model that appears often in the literature is the tubular reactor, which appears in the study of some chemical processes, see [Rut84] or [SMJ+99]. The main phenomena which takes place into the reactor are the diffusion and the convection. In order to find the model we consider the convection and diffusion of some species diluted in some neutral phase in a tubular reactor. For the sake of simplicity we consider a single species. We consider that the flow in the pipe is a steady laminar flow with constant temperature and a parabolic radial velocity profile [BSL02]. This leads to express the evolution of the mass density or equivalently the concentration of the species, known as Taylor's model of dispersion, as the following conservation law. The conserved quantity is the concentration $C(z,t)$ subject to the following balance equation:

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial z}\left(\beta_D + \beta_C\right) \tag{1.7}$$

where the flux is the sum β_D, the flux variable associated with the dispersion and β_C, the flux variable describing the convection phenomena.

The flux β_D associated with dispersion is defined by the closure equation:

$$\beta_D = -D\,\frac{\partial}{\partial z} C \tag{1.8}$$

where D denotes the dispersivity coefficient and is considered here constant (assuming that the the laminar flow is in steady state). It may be noted that this flux is exactly analogous to the heat conduction flux (1.5). The flux β_C associated with the convection is:

$$\beta_C = U\,C \tag{1.9}$$

where $U > 0$ is the constant average axial velocity of the liquid. The conservation law (1.7) with the two closure equations (1.8) and (1.9) leads to a partial differential equation of the form

$$\frac{\partial C}{\partial t}(t,z) = \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}(t,z)\right) - U\frac{\partial C}{\partial z}(t,z). \tag{1.10}$$

In a second instance let us assume that the considered species are subject to some chemical reaction. And consider a linearized chemical kinetics

$$\beta_K = -\kappa C, \tag{1.11}$$

where κ is some positive constant. This flux acts as a distributed source in the mass balance equation (1.7) due to the reaction completes the conservation law to the following balance equation :

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial z}\left(\beta_D + \beta_C\right) + \beta_K \tag{1.12}$$

Summarizing, we have the following model

$$\frac{\partial C}{\partial t}(t,z) = \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}(t,z)\right) - U\frac{\partial C}{\partial z}(t,z) - \kappa\, C(t,z). \tag{1.13}$$

*

When analyzing all these models some (fundamental) "questions" (or points) arise:

1. The first main question is that of existence and uniqueness of solutions. That is, we need to check whether the system has a solution, and whether that solution is unique. This leads, to the necessity of imposing boundary conditions on the partial differential equation (PDE) governing the system.

2. The first point leads to the question of which boundary conditions we want (or need) to impose on the system.

3. Since these systems can interact either with the environment or with other systems, we want to consider them as open systems. As such, we need to define what are the variables that the system will use to interact with other systems. Therefore, we may also want to decide which of those variables will be considered as inputs and which as outputs. Note that the interaction may also take place through the boundary of the spatial domain, and thus the boundary variables (not to confuse with boundary conditions) may also be used as interaction variables.

4. Once we have selected the inputs and outputs we can study the well-posedness properties of the resulting system. Roughly speaking, this refers to a continuity relation between the selected inputs and outputs with respect to the internal variables of the system.

5. Finally, one can proceed to study further properties of the resulting system, such as stability, controllability, and observability.

One can study these fundamental questions for each problem at hand, i.e., independently for each system. However, in this book we *do not* want to deal with these questions on a case-by-case basis. We want to look for a general structure that these models may have and exploit that structure in order to try to solve the questions above for a (possible large) class of systems. In the next section we give more details on this.

1.2. Examples Revisited

In the previous section we mentioned that we want to deal with certain classes of systems. We also mentioned that we want to do this by looking for a common structure in the models describing the dynamics of those systems. To motivate this we start by reviewing the examples presented in the previous section.

In the first three examples, i.e, Example 1.1, 1.2, and 1.3, the energy of the system can be described by a function. In addition, it can be shown that the rate of change of this energy goes via the boundary of the spatial domain. This means that there is no internal damping (or internal energy dissipation) in the system. So we start by looking for a common structure for systems that share this property. That is, we first start with systems where there is no internal energy dissipation. A common approach to start this analysis is to rewrite the model as an evolution equation. That is, as an equation of the form

$$\frac{dx}{dt}(t) = \mathcal{A}x(t), \quad x(0) = x_0, \; t \geq 0$$

where x is called the state variable and lies in the state space X, and $\mathcal{A}$ is an operator with its domain contained in X. The next example may help to clarify all this.

Example 1.6 Consider the vibrating string of Example 1.1. The energy of the system is given by

$$E(p,q) = \frac{1}{2}\int_a^b \left(\frac{1}{\rho}|p|^2 + T|q|^2\right) dz, \tag{1.14}$$

where $q(z,t) = \frac{\partial u}{\partial z}(z,t)$ is the strain and $p(z,t) = \rho\frac{\partial u}{\partial t}(z,t)$ is the momentum distribution. In order to study the properties of the system, we rewrite equation (1.1) as a first order (in time) system. One way to do this is by selecting the energy variables, i.e., p and q, as the state variables and the state space is selected as $X = L_2(a,b)^2$ with inner product $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ given by

$$\langle x, w\rangle_{\mathcal{L}} = \left\langle \begin{bmatrix} \rho^{-1}x_1 \\ T\,x_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle, \quad \forall\, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in L_2(a,b)^2.$$

Here $\langle\cdot,\cdot\rangle$ is the standard L_2-inner product, i.e., $\langle\cdot,\cdot\rangle = \int_a^b (\cdot)^T(\cdot)\,dz$. The selection of $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ as the inner product is valid since ρ and T are assumed to be positive bounded functions. Observe now that the norm on this state space becomes the energy of the system, see (1.14). Indeed, if we let $x = \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ we obtain

$$\|x\|_{\mathcal{L}}^2 = \langle x, x\rangle_{\mathcal{L}} = \left\langle \begin{bmatrix} \rho^{-1}p \\ T\,q \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix} \right\rangle = E(p,q).$$

This is the main reason for selecting $X = L_2(a,b)^2$ (with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{L}}$) as the state space, so that its norm corresponds to the expression representing the energy (in terms of the selected state variables). In this case, X is usually known as the energy state space.

Using $q(z,t) = \frac{\partial u}{\partial z}(z,t)$ and $p(z,t) = \rho\frac{\partial u}{\partial t}(z,t)$, we can rewrite the wave equation (1.1) as follows

$$\frac{\partial}{\partial t}\underbrace{\begin{bmatrix} p \\ q \end{bmatrix}}_{x}(z,t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial z}}_{\mathcal{J}}\underbrace{\begin{bmatrix} \frac{1}{\rho}p \\ Tq \end{bmatrix}}_{\mathcal{L}x}(z,t). \tag{1.15}$$

We can regard this as an evolution equation whose operator can be seen as the composition of two operators, namely $\mathcal{J}$ and $\mathcal{L}$. The operator $\mathcal{L}$ contains the parameters of the system, whereas the operator $\mathcal{J}$ can be shown to capture the internal geometric structure of the system. Now we can see that this model of the system has certain structure, that is, $\mathcal{L} = \begin{bmatrix} \rho^{-1} & 0 \\ 0 & T \end{bmatrix}$ is bounded, symmetric and positive, and the operator $\mathcal{J}$ can be written as a matrix times $\frac{\partial}{\partial z}$, which in this case is $\mathcal{J} = P_1\frac{\partial}{\partial z}$ where the matrix P_1 is symmetric. This gives immediately that the operator $\mathcal{J}$ is formally skew-adjoint, as will be seen in the next chapter. Actually, the fact that $\mathcal{J}$ is formally skew-adjoint[1] gives (assuming differentiability of x)

$$\begin{aligned}\frac{\partial}{\partial t}E = \frac{1}{2}\frac{\partial}{\partial t}\langle x, x\rangle_{\mathcal{L}} &= \frac{1}{2}\left\langle \frac{\partial x}{\partial t}, \mathcal{L}x\right\rangle + \frac{1}{2}\left\langle \mathcal{L}x, \frac{\partial x}{\partial t}\right\rangle \\ &= \frac{1}{2}\langle \mathcal{J}\mathcal{L}x, \mathcal{L}x\rangle + \frac{1}{2}\langle \mathcal{L}x, \mathcal{J}\mathcal{L}x\rangle = 0, \end{aligned} \tag{1.16}$$

where we have used (1.15). This shows that the rate of change of the energy is zero if the boundary variables are zero. This in turn, implies that there is no internal energy dissipation. Note that this last property depends only on $\mathcal{J}$ and not on $\mathcal{L}$. $*$

Remark 1.7. Note that $\mathcal{L}x = \begin{bmatrix} \frac{1}{\rho}p \\ Tq \end{bmatrix}$ equals the variational derivative, see page 14, of the energy E. The variables in $\mathcal{L}x$ are sometimes called co-energy variables since they satisfy $\langle \dot{x}, \mathcal{L}x\rangle = \frac{dE}{dt}$. Furthermore, $\mathcal{J}$ is a formally skew-adjoint operator. In this case, this differential operator corresponds to the expression of a canonical interdomain coupling between the elastic energy domain and the kinetic energy domain. This implies, by the skew-symmetry of $\mathcal{J}$, that the elastic energy is transformed into kinetic energy and viceversa, thus maintaining the total energy conserved. This is an intrinsic property of this class of skew-symmetric operators. We shall discuss more about this in the next chapter. ♣

The above example shows that based on the energy function we can obtain a model with certain structure. Later we shall show that these ideas applied to the beam equation and the suspension system lead to systems with a similar

[1] A differential operator $\mathcal{J}$ on H is *formally skew-adjoint* if it satisfies $\langle \mathcal{J}x, x\rangle_H = -\langle x, \mathcal{J}x\rangle_H$ for all x with all boundary variables set to zero.

structure. Note that there are two main advantages in doing this. One is that the norm of the state space equals the energy function. And the other is that the operator describing the evolution of the system can be split into two parts, each of them with certain structure. Furthermore, each of these operators captures different properties of the system. Also, by following the modeling process we have that one first arrives at equation (1.15) and from this the model (1.1) is obtained, see Example 7.8. So, from a modeling point of view, it seems more natural to work with model (1.15).

Remark 1.8. Readers familiar with the ideas presented in Example 1.6 will note that typically the state variables (for the wave equation) are selected as the position $u(z,t)$ and the velocity $\frac{du}{dt}(z,t)$ instead of the strain and the momentum. This leads to the selection of a state space whose inner product involves derivatives (with respect to z), see [CZ95b]. In that case one does not obtain a model with the structure described above. ♣

1.3. A class of PDE

Following the previous section we can see that it is possible to describe a class of systems by a PDE with the following structure

$$\frac{\partial x}{\partial t}(t,z) = \mathcal{J}\mathcal{L}\,x(t,z), \tag{1.17}$$

where $\mathcal{L}$ is a bounded coercive operator on $X = L_2(a,b;\mathbb{R}^n)$, and the differential operator $\mathcal{J}$ is given by

$$\mathcal{J}e = \sum_{i=0}^{N} P_i \frac{\partial^i e}{\partial z^i}, \tag{1.18}$$

with P_i, $i = \{1,2,\ldots,N\}$, constant matrices of size $n \times n$, and $x(t,z) \in \mathbb{R}^n$. Usually in applications $\mathcal{L}$ is a multiplication operator, i.e., $(\mathcal{L}\,x)(z) = \mathcal{L}(z)\,x(z)$. Furthermore, we assume that

$$P_i = (-1)^{i+1} P_i^T, \quad i = 0,1,\ldots,N. \tag{1.19}$$

The condition above implies that the differential operator $\mathcal{J}$ is formally skew-adjoint, see Chapter 2. Furthermore, we choose the norm of the state space to match the expression for the energy function which typically is described by the Hamiltonian function

$$E = \frac{1}{2}\,\langle x, \mathcal{L}\,x\rangle\,, \qquad \text{for } x \in L_2(a,b)^n.$$

Thus, from Example 1.6, we can see that the vibrating string falls into this class of systems, as well as the beam equation of Example 1.2 and the suspension system of Example 1.3. Hence, we can see that there is a variety of systems that are described by this class of PDEs.

Since $\mathcal{J}$ is skew-symmetric one obtains (formally) that the rate of change of the energy is zero, (see (1.16)). This is a property of the skew-symmetry of this operator. In fact, later it is shown that the energy preserving structure of the system is based on the operator $\mathcal{J}$. On the other hand the operator $\mathcal{L}$ captures the intrinsic properties of the system such as material properties, dimensions, and so forth.

Note however, that this class of systems does not cover Example 1.4 and the fix bed reactor of Example 1.5. In the next section we generalize this class of systems to cover those examples, and in general a larger class of systems, which also includes diffusion systems.

1.4. A class of PDE with dissipation

In the previous section we introduced a class of systems with no internal energy dissipation. In this section we consider a larger class of systems which can include this phenomena. Based on (1.17) we just add another operator that expresses the energy dissipation part of the system as follows

$$\frac{\partial x}{\partial t}(t,z) = \ (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}\, x(t,z) \tag{1.20}$$

where $\mathcal{J}$ and $\mathcal{L}$ were described in the previous section and S is a coercive operator on $L_2(a,b;\mathbb{R}^m)$. The differential operators $\mathcal{G}_R$ and its formal adjoint $\mathcal{G}_R^*$ are given by

$$\mathcal{G}_R x = \sum_{i=0}^{N} G_i \frac{\partial^i x}{\partial z^i}, \quad \mathcal{G}_R^* x = \sum_{i=0}^{N} (-1)^i G_i^T \frac{\partial^i x}{\partial z^i}, \tag{1.21}$$

where G_i, $i = \{1,2,\ldots,N\}$, are $n \times m$ constant matrices. The following example motivates the selection of this class of systems.

Example 1.9 Consider the fixed-bed reactor of Example 1.5. The system without chemical reaction is described by the PDE (1.10) and in this case we have that the skew-symmetric operator is $\mathcal{J} = -U\frac{\partial}{\partial z}$ and is associated with the convection. In a similar way as for the heat conduction, $\mathcal{G}_R = \frac{\partial}{\partial z}$ expresses both spatial derivatives related to the conservation law (1.7) and the definition of the dispersion flux (1.8). The operator $S = D$ is the dispersitivity coefficient and is positive according to the second principe of Thermodynamics. The operator $\mathcal{L}$ is simply the identity as the driving force for both phenomena may be reduced to the concentration. Once these operators are identified it is easy to see that the system is described by the PDE (1.20).

Consider now the fixed bed reactor equation (1.13). In this case we define the operator $\mathcal{G}_R$ by

$$\mathcal{G}_R = \begin{bmatrix} \frac{\partial}{\partial z} & 1 \end{bmatrix}, \quad \text{with} \quad \mathcal{G}_R^* = \begin{bmatrix} -\frac{\partial}{\partial z} \\ 1 \end{bmatrix},$$

and the symmetric operator associated with the parameters of the law of fluxes becomes

$$S = \begin{bmatrix} D & 0 \\ 0 & \kappa \end{bmatrix}.$$

*

Observe that Sturm-Liouville systems, see [NS00], are a special class of this type of equations, choose $n = m = 1$. In general, this is a large class of systems including, among others, diffusion systems as well as flexible structures with or without damping.

Since the operator $\mathcal{J}$ is assumed to be skew-symmetric and S coercive, we have that the energy of the system satisfies formally (compare with (1.16))

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\langle x, x\rangle_{\mathcal{L}} &= \frac{1}{2}\left\langle \frac{\partial x}{\partial t}, \mathcal{L}x\right\rangle + \frac{1}{2}\left\langle \mathcal{L}x, \frac{\partial x}{\partial t}\right\rangle \\ &= -\left\langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\right\rangle \le 0, \end{aligned} \tag{1.22}$$

which shows that there is energy dissipation. Note however, that equation (1.16) and (1.22) hold formally. Strictly speaking one has to consider the boundary variables, in particular, if we want to consider open systems as described in item 3 on page 5. In this case, one has

$$\frac{\partial}{\partial t}\langle x, \mathcal{L}x\rangle = -\left\langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\right\rangle + (\text{function of boundary variables}).$$

This brings us to one of the fundamental questions on page 5. How to select the boundary conditions of the system in such a way that the energy of the system (and hence the system itself) has certain behavior. Typically, it is desired that the rate of change of the energy is less than or equal to zero, i.e., $\frac{\partial}{\partial t}\langle x, \mathcal{L}x\rangle = 0$ (or ≤ 0). This behavior can also be influenced by applying an input function to the system (either through the boundary or along the spatial domain). Thus, as mentioned in item 3 on page 5, we can also consider inputs and outputs acting through the boundary of the spatial domain.

Summarizing, we want to study the fundamental questions on page 5 for a class of distributed parameter linear systems with a special structure, which occur often in applications. These systems are described by

$$\frac{\partial x}{\partial t}(t,z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}\, x(t,z), \quad x(0,z) = x_0(z), \tag{1.23a}$$

$$u(t) = \mathcal{B}x(t,z), \quad z \in (a,b),\, t \ge 0 \tag{1.23b}$$

$$y(t) = \mathcal{C}x(t,z), \tag{1.23c}$$

where $u(t)$ is the input function, $y(t)$ is the output, S and $\mathcal{L}$ are coercive operators on $L_2(a,b;\mathbb{R}^m)$ and $X = L_2(a,b;\mathbb{R}^n)$, respectively, and $\mathcal{B}$ and $\mathcal{C}$ are (boundary) operators that depend linearly on the boundary variables of x. Thus the input and output act on the boundary of the spatial domain $(a,\, b)$. The differential operators $\mathcal{J}$ and $\mathcal{G}_R$ are given by

$$\mathcal{J}x = \sum_{i=0}^{N} P_i \frac{\partial^i x}{\partial z^i}, \quad \mathcal{G}_R x = \sum_{i=0}^{N} G_i \frac{\partial^i x}{\partial z^i}, \quad \mathcal{G}_R^* x = \sum_{i=0}^{N} (-1)^i G_i^T \frac{\partial^i x}{\partial z^i}, \tag{1.24}$$

with $\mathcal{G}_R^*$ being the formal adjoint operator of $\mathcal{G}_R$ and G_i, P_i, $i = \{1, 2, \dots, N\}$, are constant real matrices of size $n \times m$, and $n \times n$, respectively. Furthermore, these matrices satisfy

$$P_i = (-1)^{i+1} P_i^T, \quad i = 0, 1, \dots, N. \tag{1.25}$$

In the next chapter we describe how to select the boundary operators $\mathcal{B}$ and $\mathcal{C}$ such that an answer to the first three points on page 5 can be given. In particular, we want to obtain a system whose energy is nonincreasing when the input is set to zero.

The motivation for considering the class of systems defined in (1.23) arises from the consideration of systems of conservation laws appearing in the modeling of physical systems, see Section 1.1 and 1.2. In the case when the differential operator consists only of the skew-symmetric term $\mathcal{J}$, i.e., $S = 0$, the system (1.23) may be related to Hamiltonian systems [Olv93] and a port-Hamiltonian formulation has been given in [vdSM02], [MvdS05], and [LZM05]. In this case, the system (1.23) corresponds to the model of a physical system where all the dissipative phenomena have been neglected. However, systems of conservation laws may of course also represent physical systems where the dissipative phenomena play an essential role as for instance the mass and heat transfer phenomena [BSL02].

Thus we use the port-Hamiltonian approach. This approach has been introduced as a geometric framework for the modeling and control of physical systems, which is based on a combination of the Hamiltonian approach and Network theory. The key idea is to associate with the energy interconnection structure a geometric object, called Dirac structure. In terms of the vibrating string example, one can see that the model is split in two parts, see (1.15). The part corresponding to the operator $\mathcal{J}$ describes the geometric structure of the system and is related to the Dirac structure. That is, $\mathcal{J}$ expresses how the internal components that comprise the system are interconnected among each other. On the other hand, the part described by $\mathcal{L}$ contains intrinsic properties of those components comprising the system. This allows the study of some properties (not all) of the system by using the (simpler) model with $\mathcal{L} = I$, see Chapter 2 for more details.

The rest of this chapter is dedicated to present some background information on the ideas that will be used later.

1.5. Boundary Control Systems (BCS)

In order to deal with the fundamental questions on page 5, we need to define a general setting on which we will be working. In this section we describe the general setting on which we try to solve the first two items of our fundamental questions. That is, from a PDE point of view we need to have existence and uniqueness of solutions, and we need to set the right boundary conditions. From a system point of view, we need to select the right variables as inputs.

In the previous sections it was mentioned that it is possible to control the behavior of the system by entering a signal through the boundary. This can be done in general for many applications. However, there are several things that need to be checked in order that the system is well formulated in certain sense. Below we clarify what we mean by a boundary control system.

In general, the class of BCS described here is based on [CZ95b, §3.3]. That is, BCS of the form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}\, x(t), \quad x(0) = x_0, \\ u(t) &= \mathfrak{B}\, x(t), \end{aligned} \tag{1.26}$$

where $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$, $u(t) \in U$, X and U separable Hilbert spaces, and the boundary operator $\mathfrak{B} : D(\mathfrak{B}) \subset X \to U$ satisfying $D(\mathfrak{A}) \subset D(\mathfrak{B})$, and

Definition 1.10. The control system (1.26) is a *boundary control system* if the following hold:

a. The operator $A : D(A) \to X$ with $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and

$$A\, x = \mathfrak{A}\, x \quad \text{for } x \in D(A)$$

is the generator of a C_0-semigroup on X.

b. There exists an $\mathfrak{R} \in \mathcal{L}(U, X)$ such that for all $u \in U$, $\mathfrak{R}u \in D(\mathfrak{A})$, the operator $\mathfrak{A}\mathfrak{R}$ is an element of $\mathcal{L}(U, X)$ and $\mathfrak{B}\mathfrak{R}u = u$ for $u \in U$. ♣

In our case, condition *a.* means that if the input is set to zero, then the resulting PDE with boundary conditions $\mathfrak{B}\, x(t) = 0$ has a unique (classical or weak) solution. Condition *b.* implies that the operator $\mathfrak{B}$ is surjective, meaning that "any" function in the input space U can be applied to the system. The operator $\mathfrak{R}$ can be considered as a right inverse of $\mathfrak{B}$.

Example 1.11 Consider the vibrating string described in Example 1.6 with the following boundary conditions

$$\frac{\partial u}{\partial t}(a, t) = 0, \quad T\frac{\partial u}{\partial z}(b, t) - \alpha\frac{\partial u}{\partial t}(b, t) = f(t), \tag{1.27}$$

where α is a positive constant and $f(t)$ is an input function. This implies that the string is clamped at the left ($z = a$) and damping is applied on the right ($z = b$). In this case we have, see (1.15), $x = \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$, $\mathfrak{A} = \mathcal{J}\mathcal{L}$ with $\frac{\partial u}{\partial t}(a) = (\rho^{-1}p)(a) = 0$,

$$\mathfrak{B}\,x = \frac{\partial u}{\partial z}(b) - \alpha\frac{\partial u}{\partial t}(b) = (Tq)(b) - \alpha(\rho^{-1}p)(b), \quad u(t) = f(t),$$

the state space is $X = L_2(a,b)^2$, and $D(\mathfrak{A}) = D(\mathfrak{B}) = H^1(a,b)^2$. In this case the semigroup generator A is given by

$$A = \mathcal{J}\mathcal{L}, \quad D(A) = \{x \in H^1(a,b)^2 \ | \ (\rho^{-1}p)(a) = 0,\ (Tq)(b) - \alpha(\rho^{-1}p)(b) = 0\}.$$

Later we show that this is a boundary control system in the sense of Definition 1.10. $*$

1.6. General notation

In this book we try to follow a standard notation that is commonly found in the literature. As usual, $\mathbb{R}$ and $\mathbb{R}_+$ denote the vector space of real and positive real numbers, respectively. $L_2(a,b,\mathbb{R}^n)$ (denoted also by $= L_2(a,b)^n$) is the standard vector space of square integrable functions on $\mathbb{R}^n$ with inner product denoted sometimes by $\langle\cdot,\cdot\rangle_{L_2(a,b,\mathbb{R}^n)}$ or $\langle\cdot,\cdot\rangle_{L_2}$. However, in this book we simply denote it by $\langle\cdot,\cdot\rangle$ when no confusion may arise. Similarly, its norm is denoted by either $\|\cdot\|_{L_2}$ or simply by $\|\cdot\|$. Also, $H^m(a,b,\mathbb{R}^n)$ or $H^m(a,b)^n$ denotes the standard Sobolev space of order m. Its inner product is denoted by $\langle\cdot,\cdot\rangle_{H^N(a,b)^n}$ or simply by $\langle\cdot,\cdot\rangle_{H^N(a,b)}$. Let Ω be an open set in $\mathbb{R}^d$. Here $\mathcal{D}(\Omega)$ is the space of all indefinitely differentiable functions with a compact support in Ω. If H is any Hilbert space, then we denote by $\langle\cdot,\cdot\rangle_H$ its inner product and by $\|\cdot\|_H$ its induced norm. By $\langle\langle\cdot,\cdot\rangle\rangle_H$ we denote the duality product between H and its dual H'. In general, for the Hilbert space $H^n = H \times \cdots \times H$ we denote, for simplicity, its inner product by either $\langle\cdot,\cdot\rangle_{H^n}$ or $\langle\cdot,\cdot\rangle_H$, its norm by either $\|\cdot\|_{H^n}$ or $\|\cdot\|_H$, and the duality product between H^n and its dual by either $\langle\langle\cdot,\cdot\rangle\rangle_{H^n}$ or simply $\langle\langle\cdot,\cdot\rangle\rangle_H$. In particular, the inner product in $\mathbb{R}^n$ is sometimes denoted by $\langle\cdot,\cdot\rangle_{\mathbb{R}}$, and similarly for its norm, i.e, $\|\cdot\|_{\mathbb{R}}$.

If X and Y are normed linear spaces, we denote by $\mathcal{L}(X,Y)$ the space of bounded linear operators from X to Y with domain equal to X. If $X = Y$ we simply write $\mathcal{L}(X)$. Similarly, we denote by $M_{n\times m}(Y)$ the set of $n \times m$ matrices with entries in the space Y and in the case $n = m$ we simply write $M_n(Y)$. We sometimes write $\mathbb{R}^{n\times m}$ in the case $Y = \mathbb{R}$. If T is a linear operator we denote by $\rho(T)$ its resolvent set and by $\sigma(T)$ its spectrum. Also $T_{|H}$ denotes the restriction of T to the space H. A self-adjoint operator, L, is *coercive* on X if there exists an $\varepsilon > 0$ such that

$$\langle Lx, x\rangle_X = \langle x, Lx\rangle_X \geq \varepsilon\,\|x\|_X^2 > 0 \qquad \text{for all } x \in D(L), \tag{1.28}$$

i.e., L has a bounded inverse. By I_X we denote the identity operator on X. However, we usually write I if it is obvious on which space is defined.

The derivative of a function $x : (a,b) \rightarrow \mathbb{R}$ with respect to the variable s is denoted by $\frac{dx}{ds}$. By $\frac{\partial w}{\partial x_1}$ we denote the partial derivative with respect to x_1. When convenient, we also use the notation $\partial_{x_1} w$. If the function w depends on time, $\dot{w}$ will also denote the time derivative. The *variational derivative* of the function $H(x)$ is the unique smooth function denoted by $\frac{\delta H}{\delta x}$ such that

$$H(x + \varepsilon\eta) = H(x) + \varepsilon \int_a^b \frac{\delta H}{\delta x} \cdot \eta \, dz + \mathrm{O}(\varepsilon^2), \tag{1.29}$$

for any $\varepsilon \in \mathbb{R}$ and any smooth function $\eta(z,t)$, see [Olv93]. For instance, consider the function

$$H(x) = \frac{1}{2} \int_a^b x^T(z) \, (\mathcal{L}\, x)(z) \, dz, \tag{1.30}$$

where $x \in L_2(a,b;\mathbb{R}^n)$ and $\mathcal{L}$ is a coercive operator on $L_2(a,b;\mathbb{R}^n)$. For $H(x)$ we have

$$\begin{aligned} H(x + \varepsilon\eta) &= \frac{1}{2} \int_a^b (x + \varepsilon\eta)^T \, \mathcal{L} \, (x + \varepsilon\eta) \, dz \\ &= \frac{1}{2} \int_a^b \left(x^T \mathcal{L}\, x + \varepsilon (x^T \mathcal{L}\, \eta + \eta^T \mathcal{L}\, x) + \varepsilon^2 \eta^T \mathcal{L}\, \eta^T \right) \, dz \\ &= H(x) + \varepsilon \int_a^b \eta^T \mathcal{L} x \, dz + \mathrm{O}(\varepsilon^2). \end{aligned} \tag{1.31}$$

From this we conclude that $\frac{\delta H}{\delta x}(x) = \mathcal{L}\, x$.

1.7. Dirac structures and port-Hamiltonian systems (PHS)

Here we give a simple definition of a Dirac structure and port-Hamiltonian systems, see for instance [vdS00], [vdSM02] or [LZM05] for a more precise definition and further details. Let $\mathcal{F}$, called the *flow space*, represent the space of rate energy variables, or in the bond-graph notation, flows. Correspondingly there exists the *effort space*, $\mathcal{E}$, which is the space of co-energy variables, or in the bond-graph notation, efforts. In the lumped-parameter finite-dimensional case the space of flows and the space of efforts simply correspond to a vector space and its dual; where the duality can be seen as 'power' duality, in the sense that the duality product of an element of the flow space with an element of the effort space results in physical power. In the distributed parameter case the space of flows $\mathcal{F}$ is an infinite-dimensional Hilbert space, and the space of efforts $\mathcal{E}$ can be defined to be (see [vdSM02]) a dual space to the space of flows, i.e., $\mathcal{E} = \mathcal{F}'$, with the duality product defined to be equal to physical power. We denote by $\langle . , . \rangle_{\mathcal{F}}$ and $\langle . , . \rangle_{\mathcal{E}}$, their corresponding inner products, respectively. Define now the *space of bond variables*, also called *bond space*, as the Hilbert space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ endowed with the natural inner product:

$$\left\langle b^1, b^2 \right\rangle_{\mathcal{B}} = \left\langle f^1, f^2 \right\rangle_{\mathcal{F}} + \left\langle e^1, e^2 \right\rangle_{\mathcal{E}}$$

for all $b^1 = \left(f^1, e^1\right) \in \mathcal{B}$, $b^2 = \left(f^2, e^2\right) \in \mathcal{B}$. In order to define a Dirac structure, we endow the bond space $\mathcal{B}$ with a *canonical symmetric pairing*, i.e., a bilinear form defined for $b^1 = \left(f^1, e^1\right)$, $b^2 = \left(f^2, e^2\right) \in \mathcal{B}$ as follows:

$$\left\langle b^1, b^2 \right\rangle_+ = \langle\langle f^1, e^2 \rangle\rangle_{\mathcal{E}} + \langle\langle f^2, e^1 \rangle\rangle_{\mathcal{E}}, \tag{1.32}$$

where $\langle\langle \cdot, \cdot \rangle\rangle_X$ denotes the duality product on $X \times X'$, i.e., $\langle\langle f, x \rangle\rangle_X = f(x)$ for $f \in X'$ and $x \in X$, where the duality can be seen as power.

We define a Dirac structure on the bond space $\mathcal{B}$ by using this canonical pairing (1.32). Denote by $\mathcal{D}^\perp$ the orthogonal subspace to $\mathcal{D}$ with respect to the symmetrical pairing (1.32):

$$\mathcal{D}^\perp = \left\{ b \in \mathcal{B} \,|\, \langle b, b' \rangle_+ = 0, \, \forall \, b' \in \mathcal{D} \right\}. \tag{1.33}$$

Definition 1.12. [vdSM02]. A *Dirac structure* $\mathcal{D}$ on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is a subspace of $\mathcal{B}$ which satisfies

$$\mathcal{D}^\perp = \mathcal{D}, \tag{1.34}$$

where the orthogonal complement is with described in (1.33). ♣

Essentially, the Dirac structure captures the natural power-conserving interconnection structure of a system since $\langle\langle f, e \rangle\rangle_{\mathcal{F}} = 0$ for all $(f, \, e) \in \mathcal{D}$.

The definition of a port-Hamiltonian system is based on the definition of two objects: the interconnection structure given by a Dirac structure and the Hamiltonian function representing the total energy of the system.

Definition 1.13. Let $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ be defined as above and consider the Dirac structure $\mathcal{D}$ and the Hamiltonian function $\mathcal{H}(x) : X \to \mathbb{R}$, where x contains the energy variables. Define the time variation of the energy variables as the flow variables, $f \in \mathcal{F}$, and the variational derivative, see (1.29), of $\mathcal{H}$ as the effort variables, $e \in \mathcal{E}$. Then the system

$$(f, \, e) = \left(\frac{dx}{dt}, \, \frac{\delta \mathcal{H}}{\delta x}(x) \right) \in \mathcal{D}, \tag{1.35}$$

is a *port-Hamiltonian system (PHS)* with total energy $\mathcal{H}$. ♣

The vibrating string described in Examples 1.1 and 1.6 is a PHS when modeled by equation (1.15), with $x = (p, q)$ being the energy variables, $\mathcal{H}$ is given by (1.14), $f = \frac{\partial x}{\partial t}$ and $e = \mathcal{L}x$ with respect to a Dirac structure induced by a (skew-symmetric) differential operator defined by $\mathcal{J}$. In fact, later we show how any skew-symmetric differential operator defines a Dirac structure, and how this Dirac structure is related to the graph of such operators. Furthermore, in the next chapters we show that the class of systems described in Section 1.4 are port-Hamiltonian systems.

Note, from Definition 1.13, that the key points in the definition of port-Hamiltonian systems are the Dirac structure and the Hamiltonian. A fundamental property in the port-Hamiltonian approach is that any power-conserving interconnection of port-Hamiltonian systems is a port-Hamiltonian system itself. In this case the interconnection of the several Dirac structures is again a Dirac structure and the total Hamiltonian is the sum of all Hamiltonians. Thus, when dealing with the interconnection of systems, we need to look for the total Dirac structure, and this together with the total Hamiltonian gives the model of the interconnected system. However, in order to interconnect a system we need to define the variables which can be used for the interconnection. These are called *port-variables*. They are again conjugate variables, i.e., variables whose product gives power. For instance, in electrical networks the port-variables are currents and voltages, and in mechanical systems we have generalized forces and velocities. In the case of the vibrating string, see Example 1.6, when modeled as a port-Hamiltonian system, see (1.15), the (boundary) port-variables are the velocity $\frac{1}{\rho}p$ and the force Tq at $z = a$ and $z = b$. Under the boundary conditions (1.27), this system can be seen as the interconnection of a vibrating string and a damper acting at $z = b$.

Another important property in the port-Hamiltonian approach is that it allows to incorporate nonlinearities that the system may have. These nonlinearities are usually included in the Hamiltonian while keeping the Dirac structure linear. This facilitate the analysis of some nonlinear systems, since some properties of the system can be checked by using the linearity of the Dirac structure. In the next two examples we show how these ideas can be applied to some nonlinear systems. Note, however, that these examples are included in order to show that the port-Hamiltonian formulation can also be used to deal with some nonlinear systems. We stress that *in this book we only deal with linear systems*.

Example 1.14 (*p*-system) This example is taken from [vdS05], see also [Eva98]. The p-system is a classical example of an infinite-dimensional port-Hamiltonian system. It corresponds to the case of two physical domains in interaction and consists of a system of two conservation laws. This system is a model for a 1-dimensional isentropic gas dynamics in Lagrangian coordinates. It is defined with the following variables: the specific volume $v(z,t) \in \mathbb{R}_+$, the velocity $u(z,t)$ and the pressure functional $p(v)$ (which is for instance in the case of a polytropic isentropic ideal gas given by $p(v) = Av^{-\gamma}$ where $\gamma \geq 1$). The p-system is then defined by the following system of partial differential equations

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial u}{\partial z} &= 0 \qquad z \in (a,b) \\ \frac{\partial u}{\partial t} + \frac{\partial p(v)}{\partial z} &= 0 \end{aligned}$$

representing the conservation of mass and of momentum. By defining the state vector as $x(z,t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v \\ u \end{bmatrix}$ and the vector valued flux $\beta(z,t) = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} -u \\ p(v) \end{bmatrix}$ the p-system is rewritten as the system of conservation laws

$$\frac{\partial x}{\partial t} + \frac{\partial \beta}{\partial z} = 0. \tag{1.36}$$

According to the framework of Irreversible Thermodynamics, the flux variables may be written as functions of the variational derivatives of some generating functionals. Consider the functional $\mathcal{H}(x) = \int_a^b H(v,u)\,dz$ where $H(v,u)$ denotes the energy density, which is given as the sum of the internal energy and the kinetic energy densities

$$H(v,u) = \mathcal{U}(v) + \frac{1}{2}u^2,$$

where $-\mathcal{U}(v)$ is a primitive function of the pressure. Note that the expression of the kinetic energy does not depend on the mass density which is assumed to be constant and for simplicity is set equal to 1. Hence no difference is made between the velocity and the momentum. The vector of fluxes β may now be expressed in term of the generating forces as follows

$$\beta = \begin{bmatrix} -\frac{\delta \mathcal{H}}{\delta u} \\ -\frac{\delta \mathcal{H}}{\delta v} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta \mathcal{H}}{\delta v} \\ \frac{\delta \mathcal{H}}{\delta u} \end{bmatrix},$$

where $\frac{\delta}{\delta w}$ represents the variational derivative with respect to the variable w, see equation (1.29). The anti-diagonal matrix represents the canonical coupling between two physical domains: the kinetic and the potential (internal) domain. The variational derivative of the total energy with respect to the state variable of one domain generates the flux variable for the other domain. Combining the equation above together with (1.36), the p-system may thus be written as the following Hamiltonian system:

$$\frac{\partial x}{\partial t} = \underbrace{\begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix}}_{\mathcal{J}} \begin{bmatrix} \frac{\delta \mathcal{H}}{\delta x_1} \\ \frac{\delta \mathcal{H}}{\delta x_2} \end{bmatrix}. \tag{1.37}$$

Note that the skew-symmetric operator $\mathcal{J}$ describing the Dirac structure is linear, and the nonlinearity is incorporated in the terms corresponding to the efforts, i.e., $\frac{\delta \mathcal{H}}{\delta x}$. $*$

Example 1.15 (Nonlinear vibrating string) It is easy to see from the previous example that the nonlinear wave equation

$$\frac{\partial^2 g}{\partial t^2} = \frac{\partial}{\partial z}\left(\sigma\left(\frac{\partial g}{\partial z}\right)\right) \qquad z \in (a,b)$$

may be expressed as a p-system by selecting the state variables (recall that the mass density is assumed to be 1) $u = \frac{\partial g}{\partial t}$, $v = \frac{\partial g}{\partial z}$, and $p(v) = -\sigma(v)$. That is, (compare with (1.15))

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix}}_{\mathcal{J}} \begin{bmatrix} \sigma(v) \\ u \end{bmatrix}. \tag{1.38}$$

This system describes the one-dimensional motion of an elastic material subjected at the stress σ, $v = \frac{\partial g}{\partial z}$ represents the displacement gradient or the strain and $u = \frac{\partial g}{\partial t}$ represents the velocity of the material. The stress-strain relation is defined by $\sigma(v)$. Hence we see that the port-Hamiltonian approach can also incorporate nonlinearities as mentioned above. In this case, the Dirac structure is linear since it is induced by the linear operator $\mathcal{J}$. The nonlinearity comes from the Hamiltonian. $*$

1.8. Dissipative systems

In this section we present a short description of dissipative systems, which is mainly based on [vdS00] and [Wil72]. For further details we refer to these two references and [Sta02].

Many important physical systems have input-output properties related to the conservation, dissipation and transport of energy. The theory surrounding such "dissipative properties" may be used as a framework for the design and analysis of control systems. The consideration of dissipativity is useful for control applications like robotics, active vibration damping and circuit theory.

In this section we consider state systems of the form

$$\Sigma : \begin{array}{ll} \dot{x} = f(x,u), & u \in U \\ y = h(x,u), & y \in Y \end{array} \tag{1.39}$$

where $x \in X$ is the state variable, X the state space, U is the input space, and Y is the output space. On the space $U \times Y$ of external variables there is defined a function

$$s := U \times Y \to \mathbb{R}, \tag{1.40}$$

called the *supply rate* and it expresses how the system interacts with the environment with respect to the inputs and outputs.

Definition 1.16. A state space system Σ is said to be *dissipative* with respect to the supply rate s if there exists a function $S : X \to \mathbb{R}_+$, called the *storage function*, such that for all $x_0 \in X$, all $t_1 \geq t_0$, and all input functions u

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), y(t))\, dt \tag{1.41}$$

where $x(t_0) = x_0$, and $x(t_1)$ is the state of Σ at time t_1 resulting from initial condition x_0 and input function $u(\cdot)$. If (1.41) holds with equality for all x_0, $t_1 \geq t_0$, and all $u(\cdot)$, then Σ is *lossless* with respect to s. ♣

Typically, the storage function is given by the energy of the system, and in that case, we say that the system is *energy preserving*

Equation (1.41) is know as the *dissipation inequality*. It expresses the relation between the change of energy in the system, i.e., $S(x(t_1)) - S(x(t_0))$ and the externally supplied energy, i.e., $\int_{t_0}^{t_1} s(u(t), y(t))\, dt$; and it means that the rate of increase of the storage cannot be larger than the supply. In other words, there cannot be internal creation of energy, only internal dissipation of energy is possible.

One important choice of supply rate is

$$s(u, y) = \langle u, y \rangle_{\mathbb{R}} = u^T y, \quad u \in U,\ y \in Y = U^*. \tag{1.42}$$

Definition 1.17. A state space system Σ with $U = Y = \mathbb{R}^n$ is *passive* (or *impedance passive* if it is dissipative with respect to the supply rate $s(u, y) = u^T y$. Σ is *strictly input passive* if there is a $\delta > 0$ such that Σ is dissipative with respect to $s(u, y) = u^T y - \delta \|u\|_{\mathbb{R}}^2$. Σ is *strictly output passive* if there exists $\varepsilon > 0$ such that Σ is dissipative with respect to $s(u, y) = u^T y - \varepsilon \|y\|_{\mathbb{R}}^2$. Finally, Σ is *impedance energy preserving* if it is lossless with respect to $s(u, y) = u^T y$. ♣

Here $\|\cdot\|_{\mathbb{R}}$, $\|\cdot\|_U$, and $\|\cdot\|_Y$ are the norms, respectively, on $\mathbb{R}^n$, U, and Y. Another second important choice of supply rate is

$$s(u, y) = \frac{1}{2}\gamma^2 \|u\|_U^2 - \|y\|_Y^2, \quad u \in U,\ y \in Y. \tag{1.43}$$

Definition 1.18. A state space system Σ is *scattering passive* if it is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2}\gamma^2 \|u\|_U^2 - \|y\|_Y^2$. Σ is *scattering energy preserving* if it is lossless with respect to $s(u, y) = \frac{1}{2}\gamma^2 \|u\|_U^2 - \|y\|_Y^2$. ♣

Following these definitions one important question arises, and that is how we may check that Σ is dissipative with respect to a given supply rate. In these book we will focus to answer this question for the class of systems introduced in Section 1.4. In the following chapters we will see how to choose the supply rate and how to obtain dissipative systems.

1.9. Main ideas and aims of this thesis

This book aims to provide a mathematical framework for the modeling and analysis of open[2] distributed parameter systems. In doing so, we follow the port-Hamiltonian approach. That is, the framework uses the port-Hamiltonian system description to express the dynamics of physical systems and their interaction with the environment. The structure of the resulting models forms the basis for the development of general analysis (and control) techniques.

From a mathematical point of view, this framework merges the approach based on Hamiltonian modeling of open distributed parameter systems, employing the notion of port-Hamiltonian systems, with the semigroup approach of infinite-dimensional systems theory.

The proposed framework can be seen as a another tool in the analysis of distributed parameter systems. The key point of the approach is the structure of the resulting model, which allows, in some cases, to provide and simplify results for classes of systems which share a similar structure of the model.

The specific aims of the book are as follows.

- To describe how a linear distributed parameter system can be represented as an infinite-dimensional port-Hamiltonian system, delineating in the process the underlying structure of the model.
- To exploit this structure in the model to study the properties of the system, e.g. well-posedness, stability, controllability, in such a way that one can analyze the essential features that are necessary to provide a starting point for a practical theory for control design in the port-Hamiltonian approach to distributed parameter systems.

1.10. Outline of the thesis

This book is divided into nine chapters. The content of the remaining chapters is briefly summarized as follows.

[2]By open system we mean a system that can interact with the environment and/or with other systems.

Chapter 2. This chapter is the starting point for the discussion in the subsequent chapters. It mainly deals with a class of boundary control systems (BCS) in one-dimensional spatial domain where the dissipation phenomena has been neglected. That is, we deal with the class of systems introduced in Section 1.3. In this chapter we answer points 1, 2, and 3 on page 5. We parameterize this class of BCS such that the resulting system is passive. This parameterization is based on the selection of two matrices that determine the input and outputs of the system. We also describe the relation of this class of BCS with the system node.

Chapter 3. In this chapter we focus on three subclasses of boundary control systems (BCS), namely, impedance energy preserving systems, scattering energy preserving systems, and output energy preserving systems. We describe the properties of their corresponding system nodes, and show that these systems are also conservative. This helps us to give some relations between observability, controllability, and stability for these subclasses of BCS.

Chapter 4. This chapter studies the Riesz basis property of a class of BCS described by first order differential operators. We show that under some common assumptions, the system has the Riesz basis property. The validity of this property results not only in the fact that the stability of the system is determined by the spectrum of the semigroup generator, but also is important since the dynamic behavior of the system can be described in the form of eigenfunction expansions of nonharmonic Fourier series.

Chapter 5. This chapter deals with stability and stabilization of the class of BCS studied in Chapters 2 and 3. We provide some results that facilitate to prove asymptotic and exponential stability of some BCS. We show that in some cases, it is possible to verify the stability property of a BCS by checking a condition on a matrix.

Chapter 6. In this chapter we extend the results presented in Chapter 2 to a larger class of system. This allows us to deal with system where the dissipation phenomena (e.g. heat transfer, damping) is present. We also study, briefly, stability properties of these class of systems.

Chapter 7. This chapter is concerned with the interconnection of systems studied in Chapters 2 and 6. It also describes how the results presented in Chapter 2 and 6 could be extended to other systems by seeing these other systems as the interconnection of systems studied in previous chapters.

Chapter 8. In this chapter we give some ideas on how the results presented in Chapter 2 could be extended to systems with d-dimensional spatial domain. We present what could be the basic calculus where the extension could be based on.

Chapter 9. This chapter contains conclusions that can be drawn from the discussion so far and highlights the contributions made in this thesis. At the end of the chapter we also present a few recommendations on possible future research directions.

Finally, we include an Appendix which briefly describes Holmgren's theorem and how we use it in this thesis. Also, the bibliography as well as an index is included.

Chapter 2

Distributed Parameter Systems Related to Skew-symmetric Operators: $1\mathrm{D}$ Case

In this chapter we deal with systems where the dissipation phenomena is neglected in the model. The results presented in the first part of this chapter are based on [LZM05]. In particular, we study systems of the form (1.23) with $S = 0$, that is

$$\frac{\partial x}{\partial t}(t,z) = \mathcal{J}\mathcal{L}x(t,z), \quad x(0,z) = x_0(z), \tag{2.1a}$$

$$u(t) = \mathcal{B}x(t,z), \quad z \in (a,b),\ t \geq 0 \tag{2.1b}$$

$$y(t) = \mathcal{C}x(t,z), \tag{2.1c}$$

where $\mathcal{B}$ and $\mathcal{C}$ are boundary operators, $\mathcal{L}$ is a bounded coercive operator on $X = L_2(a,b;\mathbb{R}^n)$, the differential operator $\mathcal{J}$ is given by

$$\mathcal{J}e = \sum_{i=0}^{N} P_i \frac{\partial^i e}{\partial z^i}, \tag{2.2}$$

with P_i, $i = \{1,2,\ldots,N\}$, constant real matrices of size $n \times n$, $e \in H^N(a,b;\mathbb{R}^n)$. Usually, in applications, $\mathcal{L}$ is a multiplication operator and thus it can be seen as a matrix whose elements depend on z. Here $H^N(a,b;\mathbb{R}^n)$ is the Sobolev space of order N, cf. [RR04]. For simplicity sometimes we will denote it by $H^N(a,b)^n$. Clearly, the operator $\mathcal{J}$ is a differential operator of order N acting on the state space $X = L_2(a,b;\mathbb{R}^n)$. The *formal adjoint* of $\mathcal{J}$ is given by (see [RR04, §5.5])

$$\mathcal{J}^*e = \sum_{i=0}^{N} (-1)^i P_i^T \frac{\partial^i e}{\partial z^i}, \quad z \in [a,b].$$

Assuming that $\mathcal{J}$ is formally skew-symmetric, i.e., $\mathcal{J}^* = -\mathcal{J}$, it follows from the above expression and (2.2) that

$$P_i = (-1)^{i+1} P_i^T, \quad i = 0, 1, \ldots, N. \tag{2.3}$$

Recall from Example 1.6, that the skew-symmetry of the operator $\mathcal{J}$ is related to conservation of energy. In fact, often this operator expresses the canonical interdomain coupling between different physical domains (e.g., elastic energy domain, kinetic energy domain) and this corresponds to a change of energy from one domain to another while keeping the total energy constant. That is why the class of systems described by (2.1) consists of systems where the dissipation phenomena has been neglected. For instance, this class of systems contains some beam equations and the well-known wave equation. This includes, in general, models which describe vibrations of flexible structures and traveling waves in acoustics.

In this chapter we explain how to select the boundary operators $\mathcal{B}$ and $\mathcal{C}$ such that the system (2.1) is a boundary control system in the sense of Section 1.5. Furthermore, by this selection of $\mathcal{B}$ and $\mathcal{C}$ the system will be dissipative (in particular, energy preserving) as explained in Section 1.8. We also see that the selection of these boundary operators is be based on the choice of a matrix, which in turn simplifies the analysis and design of this class of boundary control systems. Also, the relation with Port-Hamiltonian systems (PHS) is studied, as well as the respective Dirac structure. We start by describing the properties related to the skew-symmetric operator $\mathcal{J}$. These properties correspond to attributes coming from the internal interconnection of the elements that comprise the system. We introduce the port-variables, which are the variables that the system uses to interact with the environment. In particular, we define the boundary port-variables.

2.1. Stokes theorem and port-variables

Recall from Section 1.7 that in order to define a Dirac structure we need to introduce a symmetric pairing or a bilinear form on the so called bond space. From the same section we can see that it is not clear how to incorporate boundary variables in the definition of this bilinear form. In this section we show how to define such a symmetric form. We will also see throughout this chapter that the specification of this bilinear form is fundamental to obtain the results presented here.

We start by presenting an extension of Stokes' theorem which applies to skew-symmetric differential operators. This theorem gives rise to a Green's type identity, which in turn serves as the desired bilinear form. Thus the bilinear form arises naturally from this Stokes' theorem.

Theorem 2.1: *Let $\mathcal{J}$ be a formally skew-symmetric operator described by* (2.2). *Then for any two functions $e_1,\, e_2 \in H^N(a,b)^n$ we have*

$$\int_a^b (\mathcal{J}e_1)^T(z)\, e_2(z)\, dz + \int_a^b e_1^T(z)(\mathcal{J}e_2)(z)\, dz = \left[\begin{bmatrix} e_1^T(z), \dots & , \frac{d^{N-1}e_1^T}{dz^{N-1}}(z) \end{bmatrix} Q \begin{bmatrix} e_2(z) \\ \vdots \\ \frac{d^{N-1}e_2}{dz^{N-1}}(z) \end{bmatrix} \right]_a^b \tag{2.4}$$

where

$$Q = \begin{pmatrix} P_1 & P_2 & P_3 & \cdots & P_{N-1} & P_N \\ -P_2 & -P_3 & -P_4 & \cdots & -P_N & 0 \\ P_3 & P_4 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ (-1)^{N-1}P_N & 0 & \cdots & \cdots & & 0 \end{pmatrix}. \tag{2.5}$$

Furthermore, Q is a symmetric matrix.

PROOF: The proof is based on a iterative application of the well-know integration by parts. For the proof we refer the reader to [LZM04]. ■

Observe that the theorem above relates the integral over an interval to the boundary values. Thus we can see it as a generalization of Stokes' Theorem to skew-symmetric differential operators. Equation (2.4) can be seen as a Green identity, see [Joh78]. The above theorem also shows that any skew-symmetric differential operator gives rise to a symmetric bilinear form on the space of boundary variables, where the coefficients of the operator are captured in the matrix Q. Ideally, the bilinear form (1.32), with $f^i = (\mathcal{J}e_i)$, should not depend on the coefficients of the operator $\mathcal{J}$. In order to avoid this we introduce the boundary port-variables and a bond space in such a way that the Stokes' theorem above applied to the differential operator $\mathcal{J}$ may be expressed using the canonical symmetric pairing defined in equation (1.32). With this in mind, we first focus on some properties of the matrix Q and, based on this, we introduce some new matrices, which finally will lead us to the definition of boundary-port variables.

ASSUMPTION 2.2: Note that Q in (2.5) is nonsingular if and only if the matrix P_N is nonsingular. Thus, we assume for the rest of this chapter that Q is nonsingular. ♡

The evaluation on a and b on the right hand side of (2.4) gives rise to the following definition.

Definition 2.3. The matrix $Q_{\text{ext}} \in \mathbb{R}^{2nN \times 2nN}$ associated with the differential operator $\mathcal{J}$ is defined by

$$Q_{\text{ext}} = \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix}, \tag{2.6}$$

where Q is the symmetric matrix given in (2.5). ♣

Next we factorize Q_{ext} in such a way that it allows us to define the port-variables and at the same time the bilinear form becomes independent of the coefficients of the operator $\mathcal{J}$.

Lemma 2.4: *The matrix* $R_{\text{ext}} \in \mathbb{R}^{2nN \times 2nN}$ *defined as*

$$R_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix}, \tag{2.7}$$

is nonsingular and satisfies

$$\begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} = R_{\text{ext}}^T \, \Sigma \, R_{\text{ext}}, \tag{2.8}$$

where

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{2.9}$$

Furthermore, all possible matrices R *which satisfy* (2.8) *are given by the formula*

$$R = U \, R_{\text{ext}},$$

with U *satisfying* $U^T \Sigma \, U = \Sigma$. ♡

PROOF: The proof follows easily by putting (2.7) into (2.8). See the proof of Lemma 3.4 of [LZM05] for details. ■

Now we are in a position where we can define a proper bilinear form, which allows us to define Dirac structures and systems with certain structure. But first, based on the previous lemma, we introduce the boundary port-variables as the following linear combination of the boundary variables.

Definition 2.5. Define the *boundary trace operator* $\tau : H^N(a, b; \mathbb{R}^n) \to \mathbb{R}^{2nN}$ by

$$\tau(e) = \begin{pmatrix} e(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(b) \\ e(a) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(a) \end{pmatrix}. \tag{2.10}$$

Then, the *boundary port-variables* associated with the differential operator $\mathcal{J}$ are the vectors $e_\partial, f_\partial \in \mathbb{R}^{nN}$, defined by

$$\begin{bmatrix} f_{\partial,e} \\ e_{\partial,e} \end{bmatrix} = R_{\text{ext}}\, \tau(e), \tag{2.11}$$

where R_{ext} is defined by (2.7). ♣

We write $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ instead of $\left[\begin{smallmatrix} f_{\partial,u} \\ e_{\partial,u} \end{smallmatrix}\right]$ when the dependance on the variable u is obvious in the context. The following lemma gives more details on the trace operator.

Theorem 2.6: *Consider the boundary trace operator $\tau : H^N(a,b;\mathbb{R}^n) \to \mathbb{R}^{2nN}$ introduced in Definition 2.5. This operator is linear, bounded and surjective from $H^N(a,b;\mathbb{R}^n)$ onto $\mathbb{R}^{2nN}$, i.e.,*

$$\text{ran}\, \tau = \mathbb{R}^{2nN}.$$

PROOF: For the proof we refer to Section 7.8 of [Aub00] or to the proof of Theorem 4.5 of [LZM04]. ■

Let us stress that the definition of the boundary port-variables depends entirely on the coefficients of the operator $\mathcal{J}$, i.e., P_i. Observe that these port-variables can also be seen as an operator acting on the boundary of the spatial domain. After using the definition above and Lemma 2.4 it is easy to see that equation (2.4) becomes

$$\begin{aligned} \int_a^b (\mathcal{J}e_1)^T(z)\, e_2(z)\, dz + \int_a^b e_1^T(z)(\mathcal{J}e_2)(z)\, dz &= \begin{bmatrix} f_{\partial,e_1} \\ e_{\partial,e_1} \end{bmatrix}^T \Sigma \begin{bmatrix} f_{\partial,e_2} \\ e_{\partial,e_2} \end{bmatrix} \\ &= f_{\partial,e_1}^T e_{\partial,e_2} + e_{\partial,e_1}^T f_{\partial,e_2}, \end{aligned} \tag{2.12}$$

for $e_1,\ e_2 \in H^N(a,b)^n$. Now we can proceed to define the Dirac structure.

2.2. Dirac structure and port-Hamiltonian systems

In the previous section we showed the basic steps to choose the bilinear form needed to define the Dirac structure. Next, we need to select the flow and effort space. As mentioned earlier, the bilinear form contains elements from the boundary and elements of the state space. That is why we choose the flow and effort space as

$$\mathcal{F} = \mathcal{E} = L_2(a,b;\mathbb{R}^n) \times \mathbb{R}^{nN}, \tag{2.13}$$

with their natural inner product. It is easy to see that $\mathcal{E}$ is the dual of $\mathcal{F}$. Following Section 1.7 we define the bond space $\mathcal{B}$ as $\mathcal{F}\times\mathcal{E}$ and, based on equation (2.12), we endow $\mathcal{B}$ with the canonical symmetric pairing

$$\left\langle (f^1, f^1_\partial, e^1, e^1_\partial), (f^2, f^2_\partial, e^2, e^2_\partial) \right\rangle_+ = \left\langle f^1, e^2 \right\rangle_{L_2} + \left\langle e^1, f^2 \right\rangle_{L_2} - \left\langle f^1_\partial, e^2_{\partial,} \right\rangle_{\mathbb{R}} - \left\langle e^1_{\partial,}, f^2_\partial \right\rangle_{\mathbb{R}}, \qquad (2.14)$$

where

$$(f^i, f^i_\partial, e^i, e^i_\partial) \in \mathcal{B} \qquad i \in \{1, 2\}.$$

Let us mention again that this pairing on the bond space corresponds to the general definition given in equation (1.32). Note also that the bilinear form (2.14) is the same as equation (2.12) when we let $f = \mathcal{J}e$. Following the definition of the flow and effort space, i.e. (2.13), one immediately sees that every element of these spaces is a vector, with the top part being a function (which lies in the state space $X = L_2(a, b; \mathbb{R}^n)$), and the bottom element being part of the (boundary) port-variable.

Now we have all necessary ingredients to define the Dirac structure associated with the skew-symmetric operator $\mathcal{J}$ given by (2.2)–(2.3).

Theorem 2.7: *Consider the skew-symmetric operator $\mathcal{J}$ given by* (2.2)–(2.3) *together with the boundary port-variables as described in Definition* 2.5*. Then, the subspace $D_\mathcal{J}$ of $\mathcal{B}$ defined by*

$$D_\mathcal{J} = \left\{ \begin{bmatrix} f \\ f_\partial \\ e \\ e_\partial \end{bmatrix} \in \mathcal{B} \;\middle|\; e \in H^N(a, b; \mathbb{R}^n),\ \mathcal{J}e = f,\ \begin{bmatrix} f_{\partial,e} \\ e_{\partial,e} \end{bmatrix} = R_{\text{ext}}\, \tau(e) \right\} \qquad (2.15)$$

is a Dirac structure with respect to the bilinear form (2.14).

PROOF: The proof is based on Definition 1.12. First one proves that for any two elements lying in the subspace $D_\mathcal{J}$, say b_1, b_2, there holds $\langle b_1, b_2 \rangle_+ = 0$, which follows immediately from (2.12). This gives $D_\mathcal{J} \subset D_\mathcal{J}^\perp$. Obviously the other part is to prove the other inclusion. For the proof see [LZM04, §3]. ■

It is worth mentioning that the Dirac structure as a subspace of the bond space $\mathcal{B}$ is closed. In fact, it can be seen as the graph of a skew-adjoint operator, which is related to $\mathcal{J}$, see Theorem 3.12.

Observe that we have not defined dynamics, we have just defined the geometric structure of the system (2.1). The dynamics are defined when we define the corresponding port-Hamiltonian system. To do so, we need the energy function. Assume that the Hamiltonian can be expressed as

$$H(x) = \frac{1}{2}\int_a^b x^T(z)\,(\mathcal{L}\,x)(z)\,dz, \tag{2.16}$$

where x is the energy variable and $\mathcal{L}$ is the coercive operator described at the beginning of this chapter. Then we can adapt the definition of port-Hamiltonian systems (PHS), see Definition 1.13, to include boundary port-variables as follows

- define the time variation of the energy variables as the flow variables, i.e., $f = \frac{\partial x}{\partial t}$,
- define the variational derivative of H as the effort variables, $e = \frac{\delta H}{\delta x} = \mathcal{L}\,x$, and
- include the two external boundary port-variables f_∂, e_∂.

Then the system

$$\begin{bmatrix} f \\ f_\partial \\ e \\ e_\partial \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial t} \\ f_\partial \\ \mathcal{L}x \\ e_\partial \end{bmatrix} \in D_{\mathcal{J}} \tag{2.17}$$

is a port-Hamiltonian system with total energy H. It is easy to see from Theorem 2.7 that the condition above implies that

$$f = \mathcal{J}e \qquad \text{and thus} \qquad \frac{\partial x}{\partial t} = \mathcal{J}\mathcal{L}\,x,$$

which is the same equation that defines our class of systems, see (2.1a). Furthermore, since $(f,\, f_\partial,\, e,\, e_\partial)$ lies in the Dirac structure $D_{\mathcal{J}}$, we must have from Definition 1.12 that any two trajectories lying in the Dirac structure, say $(f^i,\, f^i_\partial,\, e^i,\, e^i_\partial)$ for $i = \{1, 2\}$, satisfy for each time instant t

$$\begin{aligned} 0 &= \left\langle (f^1, f^1_\partial, e^1, e^1_\partial), (f^2, f^2_\partial, e^2, e^2_\partial) \right\rangle_+ = \left\langle (\mathcal{J}e^1, f^1_\partial, e^1, e^1_\partial), (\mathcal{J}e^2, f^2_\partial, e^2, e^2_\partial) \right\rangle_+ \\ &= \left\langle \mathcal{J}e^1, e^2 \right\rangle_{L_2} + \left\langle e^1, \mathcal{J}e^2 \right\rangle_{L_2} - \langle f^1_\partial, e^2_{\partial,} \rangle_{\mathbb{R}} - \langle e^1_{\partial,}, f^2_\partial \rangle_{\mathbb{R}} \\ &= \left\langle \mathcal{J}e^1, e^2 \right\rangle_{L_2} + \left\langle e^1, \mathcal{J}e^2 \right\rangle_{L_2} - \left\langle \begin{bmatrix} f^1_\partial \\ e^1_\partial \end{bmatrix}, \Sigma \begin{bmatrix} f^2_\partial \\ e^2_\partial \end{bmatrix} \right\rangle_{\mathbb{R}}. \end{aligned} \tag{2.18}$$

Hence, any two trajectories lying on the Dirac structure satisfy equation (2.12) pointwise in time and, therefore, they also satisfy Theorem 2.1. This property is used very often since it is important when defining and studying properties of the class of boundary control systems (2.1). At this stage it is significant to mention two points.

- The input and output variables, i.e., u and y, as appear in (2.1) are selected as a linear combination of the port-variables. Note that $e = \mathcal{L}x$ and thus we infer from (2.1) that the boundary operator $\mathcal{B}$ will contain an operator (to be selected) times $R_{\text{ext}}\,\tau(\cdot)$. Hence, in equation (2.12) the term corresponding to the port-variables can be replaced by a term depending only on the input u and output y.
- Observe that the time variation of the energy H in (2.16) is (assuming differentiability)

$$\frac{dH}{dt} = \int_a^b \left(\frac{\partial x}{\partial t}(z)\right)^T (\mathcal{L}\,x)(z)\,dz = \int_a^b f^T\,e\,dz = \int_a^b (\mathcal{J}e)^T\;e\,dz,$$

  which can be written in terms of the port-variables by (2.12). This in turn, implies by the preceding item that $\frac{dH}{dt}$ can be written in terms of the inputs and outputs. Hence, obtaining dissipative (in this case lossless) systems in the sense of Section 1.8 with the storage function H.

In the next section we show how to select input and outputs, such that the system is a boundary control system in the sense of Section 1.5 and dissipative in the sense of Section 1.8.

2.3. Parametrization of boundary control systems

In the previous section we have associated to the skew-symmetric operator $\mathcal{J}$ a Dirac structure $\mathcal{D}_{\mathcal{J}}$. In this section we define dynamic systems with inputs, states, and outputs with respect to this Dirac structure. These systems are boundary control systems in the sense of Section 1.5, which implies that the controls and observations act on the boundary of the spatial domain. With respect to the Dirac structure $\mathcal{D}_{\mathcal{J}}$ it is possible to define many systems. However, we only consider those systems for which the energy does not grow when the input is zero, i.e., dissipative systems. This implies that the associated semigroup is a contraction. Again, the results presented in this section are based on [LZM05] and [LZM04].

We begin by showing that $\mathcal{J}$ is the infinitesimal generator of a contraction semigroup for appropriate choices of the boundary conditions (inputs).

2.3.1. Contraction semigroups associated with $\mathcal{J}$

We begin by studying the differential operator $\mathcal{J}$ for different boundary conditions (inputs). As stated above we want to characterize those boundary conditions for which the associated differential operator is the infinitesimal generator of a strongly continuous semigroup. In [LZM05] and [LZM04] the authors based this parametrization on the selection of a full rank matrix W of size $nN \times 2nN$ satisfying

$$W\,\Sigma\,W^T \geq 0, \tag{2.19}$$

where $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ was defined in Lemma 2.4. In those papers it is also shown that the condition above holds if and only if W can be parameterized as

$$W = S \left[\begin{array}{cc} I+V, & I-V \end{array}\right], \qquad V\,V^T \leq I, \tag{2.20}$$

where $S \in \mathbb{R}^{nN \times nN}$ is a nonsingular matrix and $V \in \mathbb{R}^{nN \times nN}$ is clearly a contraction matrix, see [LZM05, Lemma A.1]. It is worth mentioning that, under these conditions, the kernel of W satisfies

$$\ker W = \operatorname{ran} \left[\begin{array}{c} I-V \\ -(I+V) \end{array}\right]. \tag{2.21}$$

In the following theorem it is shown that if the port-variables are restricted to the kernel of W, then this defines the domain of a contraction semigroup associated with the operator $\mathcal{J}$.

Theorem 2.8: *Let W be a full rank real matrix of size $nN \times 2nN$ and consider the skew-symmetric operator $\mathcal{J}$ described by* (2.2)–(2.3) *together with the boundary port-variables as described in Definition* 2.5. *Define the operator $\mathcal{A}$ and its domain, $D(\mathcal{A})$, as*

$$\mathcal{A}\,e = \mathcal{J}\,e \tag{2.22}$$

and

$$D(\mathcal{A}) = \left\{ e \in H^N(a,b;\mathbb{R}^n) \;\middle|\; \left[\begin{array}{c} f_{\partial,e} \\ e_{\partial,e} \end{array}\right] \in \ker W \right\}. \tag{2.23}$$

Then $\mathcal{A}$ generates a contraction semigroup $T(t)$, $t \geq 0$, on $L_2(a,b;\mathbb{R}^n)$ if and only if W satisfies $W\,\Sigma\,W^T \geq 0$.

Furthermore, $\mathcal{A}$ is the infinitesimal generator of a unitary semigroup on $L_2(a,b;\mathbb{R}^n)$ if and only if W satisfies $W\,\Sigma\,W^T = 0$.

PROOF: The proof is based on [GG91, Theorem 3.1.6], which in turn depends on proving the validity of an equality of the type (2.18). For details see [LZM05, §4.1]. ∎

Remark 2.9. The operator $\mathcal{A}$ can also be written in terms of the Dirac structure

$\mathcal{D}_{\mathcal{J}}$ as $\mathcal{A}\, e = \mathcal{J}\, e$ with domain

$$D(\mathcal{A}) = \Big\{ e \in L_2(a,b;\mathbb{R}^n) \ \mid \ \text{the boundary port-variable associated to } e, \text{ i.e.} \left[\begin{array}{c} f_{\partial,e} \\ e_{\partial,e} \end{array}\right], \text{ is in } \ker W \text{ and there exists an } f \in L_2(a,b;\mathbb{R}^n) \text{ such that } (f,\ f_{\partial,e},\ e,\ e_{\partial,e}) \in \mathcal{D}_{\mathcal{J}} \Big\},$$

where $\mathcal{D}_{\mathcal{J}}$ is described in Theorem 2.7. ♣

In the previous section it was mentioned that the boundary operator $\mathcal{B}$ appearing in (2.1b) could be chosen as an operator times $R_{\text{ext}}\,\tau(\cdot)$. Following the theorem above, we can deduce that it can be selected as $\mathcal{B}(\cdot) = W\, R_{\text{ext}}\tau(\cdot)$, where $\tau(\cdot)$ is the boundary evaluation as given in Definition 2.5. Hence, we have parameterized the set of boundary conditions for which the partial differential equation (2.1) with input $u = 0$ has a unique (strong or weak) solution, see [CZ95b]. Moreover, it is a boundary control system in the sense of Section 1.5 provided that $\mathcal{B}$ is surjective. Note, in this case, that the point (*a*) in Definition 1.10 is true if the conditions in Theorem 2.8 are satisfied. Thus we only need to prove point (*b*) in that definition, that is, to prove that the operator $\mathcal{B}(\cdot) = W\, R_{\text{ext}}\tau(\cdot)$ is surjective. But this follows easily from Theorem 2.6. In the next section we will give more details on this as well as how to select the output so that the system is dissipative.

2.3.2. Boundary control systems associated with $\mathcal{J}$

In the previous subsection we have derived the family of contraction semigroups associated with a skew-symmetric differential operator $\mathcal{J}$. We showed that this family originate from the Dirac structure $\mathcal{D}_{\mathcal{J}}$ associated with $\mathcal{J}$. More precisely, we have parameterized these semigroups by a family of subspaces of the boundary port-variables, defined as the kernel of a class of matrices W. In the following theorem, which is taken from [LZM05], the authors use this W to define boundary inputs/controls. Since the rank of W is nN and since we have $2nN$ boundary variables, it is clear that we use half of the set of boundary variables to define inputs. We show that the other half may be regarded as outputs.

Theorem 2.10: *Let W be a full rank real matrix of size $nN \times 2nN$ and consider the skew-symmetric operator $\mathcal{J}$ described in Theorem 2.7. If W has full rank and satisfies $W\Sigma W^T \geq 0$, where Σ is defined in* (2.9)*, then the following system*

$$\frac{\partial x}{\partial t}(t) = \mathcal{J} x(t), \quad \textit{or equivalently (see (2.17))} \quad \Big(\dot{x}(t),\ f_{\partial}(t),\ x(t),\ e_{\partial}(t)\Big) \in \mathcal{D}_{\mathcal{J}} \tag{2.24}$$

defined on the state space $L_2(a,b;\mathbb{R}^n)$ with input

$$u(t) = \mathcal{B}\,x(t) = W \begin{bmatrix} f_{\partial,x}(t) \\ e_{\partial,x}(t) \end{bmatrix} \tag{2.25}$$

is a boundary control system on $L_2(a,b;\mathbb{R}^n)$. Furthermore, the operator $\mathcal{A} = \mathcal{J}$ with domain

$$D(\mathcal{A}) = \left\{ x \in H^N(a,b;\mathbb{R}^n) \;\middle|\; \begin{bmatrix} f_{\partial,x} \\ e_{\partial,x} \end{bmatrix} \in \ker W \right\}, \tag{2.26}$$

generates a contraction semigroup on $L_2(a,b;\mathbb{R}^n)$.

Let $\widetilde{W}$ be a full rank matrix of size $nN \times 2nN$ such that $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ is invertible. If we define the linear mapping $\mathcal{C} : H^N(a,b;\mathbb{R}^n) \to \mathbb{R}^{nN}$ as,

$$\mathcal{C}x(t) := \widetilde{W} \begin{bmatrix} f_{\partial,x}(t) \\ e_{\partial,x}(t) \end{bmatrix} \tag{2.27}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{2.28}$$

then for $u \in C^2(0,\infty;\mathbb{R}^{nN})$, $x(0) \in H^N(a,b;\mathbb{R}^n)$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{L_2} = \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{2.29}$$

where

$$P^{-1}_{W,\tilde{W}} = \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix} \Sigma \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^T = \begin{bmatrix} W\Sigma W^T & W\Sigma \widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{bmatrix}. \tag{2.30}$$

Furthermore, the invertibility of the matrix $\left(\begin{smallmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{smallmatrix}\right)$ is equivalent to the invertibility of $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$.

PROOF: We give a sketch of the proof, for details see [LZM05, §4]. That we have a boundary control system in the sense of Definition 1.10 follows from Theorem 2.8 (where the semigroup generator property is proven) and Theorem 2.6 (from which the existence of the operator $\mathcal{R}$ follows easily, since W has full row-rank). By the definition of $\mathfrak{R}$ and $D(\mathcal{A})$, we see that the conditions stated in the theorem are the same as $x(0) - \mathfrak{R}u(0) \in D(\mathcal{A})$. Hence by Theorem 3.3.3 of [CZ95b] we have that there exists a classical solution of (2.24)–(2.25). Hence, in particular, $x(t) \in H^N(a,b;\mathbb{R}^n)$ holds pointwise in t, $x(t)$ is differentiable as a function of t, and $\dot{x}(t) = \mathcal{J}x(t)$. Using this, we obtain

$$\begin{aligned}\frac{d}{dt}\|x(t)\|^2_{L_2} &= \frac{d}{dt}\langle x(t), x(t)\rangle_{L_2} = \langle \dot{x}(t), x(t)\rangle_{L_2} + \langle x(t), \dot{x}(t)\rangle_{L_2} \\ &= \langle \mathcal{J}x(t), x(t)\rangle_{L_2} + \langle x(t), \mathcal{J}x(t)\rangle_{L_2} \\ &= \begin{bmatrix} f_{\partial,x}(t) \\ e_{\partial,x}(t) \end{bmatrix}^T \Sigma \begin{bmatrix} f_{\partial,x}(t) \\ e_{\partial,x}(t) \end{bmatrix}, \end{aligned} \tag{2.31}$$

and that the relation between the port-variables and the input-output is given by

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix} \begin{bmatrix} f_{\partial,x} \\ e_{\partial,x} \end{bmatrix},$$

where $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ is nonsingular. See [LZM05, §4] or [LZM04] for more details. ∎

Remark 2.11. Note that for the same Dirac structure the properties of the PDE obtained by a choice of inputs and outputs can be completely different. Hence for the same underlying Dirac structure, many different systems theoretic properties are possible. ♣

Typically, for linear boundary control systems, the norm of the state space is selected to match the energy of the system. In that case, equation (2.29) represents an energy balance equation. This in turn, implies that the system is lossless or energy preserving, see Section 1.8. Also, note that the matrix $P_{W,\tilde{W}}$ appearing in equation (2.29) depends entirely on the matrices W and $\widetilde{W}$ and it determines the supply rate of the obtained passive system. Thus, we could, for instance, select those matrices so that the resulting system is either impedance energy preserving or scattering energy preserving. Later we shall give more details on this.

Comparing the system described in Theorem 2.10 with the system appearing in (2.1) we immediately see that we have not included the operator $\mathcal{L}$. In the next section we show that once we have proved existence of solutions for systems associated to the skew-symmetric operator $\mathcal{J}$ (with $\mathcal{L} = I$), i.e., Theorem 2.8, the same result is easily extended to systems which include the operator $\mathcal{L}$, i.e. (2.1). This shows one of the advantages of using the port-Hamiltonian approach, since we can study some properties of the system associated with $\mathcal{J}$ and from there conclude the same results for the whole system. In other words, we have split the model of the system into two parts: The part related to $\mathcal{J}$, which expresses the geometric interconnection structure, and the part related to $\mathcal{L}$, which represents intrinsic properties of the system (like variable or constant parameters).

Example 2.12 Consider the vibrating string of Example 1.1 described by (1.15) with $\rho = T = 1$, i.e.,

$$\frac{\partial}{\partial t}\underbrace{\begin{bmatrix} p \\ q \end{bmatrix}}_{x}(z,t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P_1}\frac{\partial}{\partial z}\underbrace{\begin{bmatrix} p \\ q \end{bmatrix}}_{x}(z,t),$$

where $x = \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ contains the energy variables. From the equation above we can distinguish that the operator $\mathcal{J}$ is, in this case, described by $\mathcal{J} = P_1 \frac{\partial}{\partial z}$. The assumption $\rho = T = 1$ implies that the momentum p and the velocity p/ρ can be considered to be the same; similarly for the strain q and the stress Tq. In this case we have that the boundary port-variables are, see (2.7) and Definition 2.5,

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} x(b) \\ x(a) \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} q(b)-q(a) \\ p(b)-p(a) \\ p(b)+p(a) \\ q(b)+q(a) \end{bmatrix} = \begin{bmatrix} f_{\partial_1} \\ f_{\partial_2} \\ e_{\partial_1} \\ e_{\partial_2} \end{bmatrix}. \tag{2.32}$$

It is important to notice that

$$\|x\|^2 = \int_a^b |x(z)|^2 \, dz = \int_a^b \left\| \begin{bmatrix} p(z) \\ q(z) \end{bmatrix} \right\|^2 dz = \int_a^b |p(z)|^2 + |q(z)|^2 \, dz,$$

which is the same expression that define the energy of the system, see (1.14). There are different settings to which the string can be subjected to. This corresponds to different boundary conditions, for instance

$p(a,t) = 0,\ p(b,t) = 0$	string clamped at both ends (clamped-clamped)
$p(a,t) = 0,\ q(b,t) = 0$	string clamped at $z = a$, free at $z = b$ (clamped-free),

are two examples of possible boundary conditions.

- (clamped-clamped) In our setting the first case can be obtained by selecting the matrix W

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

and letting the input $u(t) = 0$, $t \geq 0$. Indeed, in this case we have that

$$u = W \begin{bmatrix} f_{\partial,x} \\ e_{\partial,x} \end{bmatrix} = \begin{bmatrix} p(a) \\ -p(b) \end{bmatrix} \quad \text{and} \quad W\Sigma W^T = 0.$$

This guarantees, by Theorem 2.8, that this system has a solution. In fact, the operator given by

$$\mathcal{A}x = P_1 \frac{\partial x}{\partial z}, \quad D(\mathcal{A}) = \left\{ x \in H^1(a,b)^2 \mid W \begin{bmatrix} f_{\partial,x} \\ e_{\partial,x} \end{bmatrix} = 0 \right\},$$

with $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ given in (2.32), generates a unitary semigroup. A logical selection for the output will be to observe the stress (force) at both ends, i.e., $y = \left[\begin{smallmatrix} -q(a) \\ -q(b) \end{smallmatrix}\right]$. This corresponds to a matrix

$$\widetilde{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

Then, from equation (2.30) we obtain that

$$P_{W,\tilde{W}}^{-1} = P_{W,\tilde{W}} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

which, in turn gives from (2.29) that

$$\begin{aligned} \frac{d}{dt}E(t) = \frac{1}{2}\frac{d}{dt}\|x(t)\|^2 &= \frac{1}{2}\begin{bmatrix} u^T(t) & y^T(t) \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \\ &= u^T(t)\,y(t) = 0, \end{aligned}$$

since we took $u(t) = 0$. $*$

Observe from the previous example that when the operator $\mathcal{L}$ is taken into account, the norm on the state space has to be changed so that it matches the energy of the system. This will be the main idea in the argument used in the next section.

2.3.3. Boundary control systems associated with $\mathcal{J}\mathcal{L}$

In the previous section we defined boundary control systems associated with the skew-symmetric operator $\mathcal{J}$. We showed how to parameterize the selection of inputs and outputs in terms of matrices. However, we did not completely characterize the selection of inputs and outputs for the class of systems (2.1) since we did not include the coercive operator $\mathcal{L}$. Nevertheless, at the end of Section 2.2, we outlined how this could be done. Also, at the end of the previous section we mentioned that this could be done by modifying the norm on the state space.

In Section 2.3.2 (when $\mathcal{L}$ was assumed to be the identity operator) we considered the state space $X = L_2(a, b; \mathbb{R}^n)$ with its natural inner product, i.e.,

$$\langle x_1, x_2 \rangle = \int_a^b x_1^T(z)\, x_2(z)\, dz.$$

In order to deal with the class of systems (2.1) when $\mathcal{L}$ is taken to be any bounded coercive operator we need to redefine this state space as follows. Let the *energy state space* be defined by

$$\begin{aligned} &X = L_2(a, b; \mathbb{R}^n) \text{ with inner product } \langle x_1, x_2 \rangle_{\mathcal{L}} = \langle x_1, \mathcal{L}x_2 \rangle \\ &\text{and corresponding norm } \|x_1\|_{\mathcal{L}}^2 = \langle x_1, x_1 \rangle_{\mathcal{L}} \text{ for any } x_1,\ x_2 \in X, \end{aligned} \tag{2.33}$$

where $\langle \cdot, \cdot \rangle$ is the natural L_2-inner product. Since $\mathcal{L}$ is assumed to be a bounded coercive operator, it is easy to see that the natural norm on X and the $\mathcal{L}$-norm are equivalent. Also, observe that this norm corresponds to a Hamiltonian of the form (2.16), i.e.,

$$H(x) = \frac{1}{2}\int_a^b x^T(z)\,(\mathcal{L}\,x)(z)\,dz = \frac{1}{2}\,\|x\|_{\mathcal{L}}^2\,. \tag{2.34}$$

Now we can proceed to adapt the definition of port-Hamiltonian systems (PHS), see Definition 1.13, to include boundary port-variables as follows (see the argument after equation (2.16))

- define the time variation of the energy variables as the flow variables, i.e., $f = \frac{\partial x}{\partial t}$,
- define the variational derivative of H as the effort variables, $e = \frac{\delta H}{\delta x} = \mathcal{L}\,x$,
- include the two external boundary port-variables f_∂, e_∂.

Then the system

$$\begin{bmatrix} f \\ f_\partial \\ e \\ e_\partial \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial t} \\ f_\partial \\ \mathcal{L}x \\ e_\partial \end{bmatrix} \in D_{\mathcal{J}}$$

is a port-Hamiltonian system with total energy H. From Theorem 2.7 it is easy to see that the conditions above gives that

$$f = \mathcal{J}e \qquad \text{and thus} \qquad \frac{\partial x}{\partial t} = \mathcal{J}\mathcal{L}\,x,$$

which is the same equation that defines our class of systems, see (2.1a). Furthermore, since $(f,\, f_\partial,\, e,\, e_\partial)$ lies in the Dirac structure $D_{\mathcal{J}}$, we must have from Definition 1.12 that any two trajectories lying in the Dirac structure, say $(f^i,\, f^i_\partial,\, e^i,\, e^i_\partial)$ for $i = \{1, 2\}$, satisfy for each time instant t

$$\begin{aligned} 0 &= \left\langle (f^1, f^1_\partial, e^1, e^1_\partial), (f^2, f^2_\partial, e^2, e^2_\partial) \right\rangle_+ = \left\langle (\mathcal{J}e^1, f^1_\partial, e^1, e^1_\partial), (\mathcal{J}e^2, f^2_\partial, e^2, e^2_\partial) \right\rangle_+ \\ &= \left\langle \mathcal{J}e^1, e^2 \right\rangle_{L_2} + \left\langle e^1, \mathcal{J}e^2 \right\rangle_{L_2} - \left\langle f^1_\partial, e^2_{\partial,} \right\rangle_{\mathbb{R}} - \left\langle e^1_{\partial,}, f^2_\partial \right\rangle_{\mathbb{R}} \\ &= \langle \mathcal{J}\mathcal{L}x_1, \mathcal{L}x_2 \rangle_{L_2} + \langle \mathcal{L}x_1, \mathcal{J}\mathcal{L}x_2 \rangle_{L_2} - \left\langle f^1_\partial, e^2_{\partial,} \right\rangle_{\mathbb{R}} - \left\langle e^1_{\partial,}, f^2_\partial \right\rangle_{\mathbb{R}} \\ &= \langle \mathcal{J}\mathcal{L}x_1, x_2 \rangle_{\mathcal{L}} + \langle x_1, \mathcal{J}\mathcal{L}x_2 \rangle_{\mathcal{L}} - \left\langle f^1_\partial, e^2_{\partial,} \right\rangle_{\mathbb{R}} - \left\langle e^1_{\partial,}, f^2_\partial \right\rangle_{\mathbb{R}}\,. \end{aligned} \tag{2.35}$$

All this, together with the expression for the Dirac structure (2.15) shows that, in this case, the definition of PHS corresponds, in part, to the abstract system $\dot{x}(t) = \mathcal{A}x(t)$ where the differential operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \mathcal{J}\mathcal{L},$$

which need not be skew-symmetric nor need have constant coefficients. Note that so far we have defined linear port-Hamiltonian systems with boundary port-variables using the definition of Dirac structure for which the port-variables are not split into input and output variables. However, we have seen in Section 2.3.2 that using specific subspaces of the port-variables, one may define input and output variables as belonging to complementary subspaces of the boundary port-variables. Moreover, by choosing appropriately these subspaces, one may define a boundary control system with its associated semigroup being a contraction. In the sequel we reformulate the boundary port-Hamiltonian system as a boundary control system. We use the parametrization of the input and output variables and the contractive semigroups associated with the Dirac structure $\mathcal{D}_{\mathcal{J}}$ given in Section 2.3.2. The state variables have become the image of the effort variables through the coercive operator $\mathcal{L}^{-1}$. In this case, the domain of the differential operator $\mathcal{J}\mathcal{L}$ becomes

$$D(\mathcal{J}\mathcal{L}) = \left\{x \in X \mid \mathcal{L}\,x \in H^N(a,b;\mathbb{R}^n)\right\}. \tag{2.36}$$

First we prove the equivalent of Theorem 2.8 and then we formulate the boundary control systems.

Theorem 2.13: *Let W be a full rank real matrix of size $nN \times 2nN$ and consider the operator $\mathcal{J}\mathcal{L}$ as described above on the state space X given by* (2.33)*. Define the operator $\mathcal{A}_{\mathcal{L}}$ and its domain, $D(\mathcal{A}_{\mathcal{L}})$, as*

$$\mathcal{A}_{\mathcal{L}}\,x = \mathcal{J}\mathcal{L}\,x \tag{2.37}$$

and

$$D(\mathcal{A}_{\mathcal{L}}) = \left\{x \in X \mid \mathcal{L}\,x \in H^N(a,b;\mathbb{R}^n),\ \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} \in \ker W \right\}. \tag{2.38}$$

Then $\mathcal{A}_{\mathcal{L}}$ generates a contraction semigroup $T(t)$, $t \geq 0$, on X if and only if W satisfies $W\,\Sigma\,W^T \geq 0$.

Furthermore, $\mathcal{A}_{\mathcal{L}}$ is the infinitesimal generator of a unitary semigroup on $L_2(a,b;\mathbb{R}^n)$ if and only if W satisfies $W\,\Sigma\,W^T = 0$.

PROOF: We use the Lümer-Phillips theorem, see [Paz83]. This theorem states that an operator, say A, on a space H generates a contraction semigroup if and only if it satisfies $\langle Ax, x\rangle_H \leq 0$ for all $x \in D(A)$ and $\langle A^*y, y\rangle_H \leq 0$ for all $y \in D(A^*)$.

First notice that the domain of $\mathcal{A}_{\mathcal{L}}$ is equal to $D(\mathcal{A}_{\mathcal{L}}) = \{x \in X \mid \mathcal{L}x \in D(\mathcal{A})\}$ where $\mathcal{A}$ is described in Theorem 2.8. For $x \in D(\mathcal{A}_{\mathcal{L}})$ we have

$$\langle \mathcal{A}_{\mathcal{L}}x, x\rangle_{\mathcal{L}} = \langle \mathcal{J}\mathcal{L}x, \mathcal{L}x\rangle = \langle \mathcal{J}e, e\rangle,$$

where $e = \mathcal{L}x$. Since $e \in D(\mathcal{A})$, see (2.23), where $\mathcal{A} = \mathcal{J}$ is the operator in Theorem 2.8, we conclude that

$$\langle \mathcal{A}_{\mathcal{L}} x, x \rangle_{\mathcal{L}} = \langle \mathcal{A} e, e \rangle ,$$

which is non-positive, since $\mathcal{A}$ generates a contraction semigroup on $L_2(a,b)^n$, see Theorem 2.8.

Next we show that $\langle \mathcal{A}^*_{\mathcal{L}} y, y \rangle_{\mathcal{L}} \leq 0$ on $D(\mathcal{A}^*_{\mathcal{L}})$. We know from (2.31) that (recall that $\langle \cdot, \cdot \rangle_{L_2}$ is a real inner product and thus $\langle e_1, \mathcal{J} e_2 \rangle_{L_2} = \langle \mathcal{J} e_2, e_1 \rangle_{L_2}$)

$$\langle e, \mathcal{J} e \rangle_{L_2} = \frac{1}{2} \begin{bmatrix} f_{\partial,e} \\ e_{\partial,e} \end{bmatrix}^T \Sigma \begin{bmatrix} f_{\partial,e} \\ e_{\partial,e} \end{bmatrix}. \tag{2.39}$$

Since $\langle \mathcal{A}_{\mathcal{L}}\, x, y \rangle_{\mathcal{L}} = \langle \mathcal{J}\mathcal{L}\, x, \mathcal{L} y \rangle$ for any $x \in D(\mathcal{A}_{\mathcal{L}})$, it is not hard to show that $\mathcal{A}^*_{\mathcal{L}} = \mathcal{A}^* \mathcal{L}$ with $D(\mathcal{A}^*_{\mathcal{L}}) = \{ y \in X \mid \mathcal{L} y \in D(\mathcal{A}^*) \}$. Using a similar argument as above we can find that $\langle \mathcal{A}^* y, y \rangle_{\mathcal{L}} \leq 0$ on $D(\mathcal{A}^*)$. This completes the proof. ■

Observe that once we have proved existence of solutions for the associated operator $\mathcal{J}$, the same results follows easily for the operator $\mathcal{J}\mathcal{L}$, as mentioned earlier. However, it is worth mentioning that the semigroup generated by $\mathcal{A}_{\mathcal{L}}$, say $T_{\mathcal{L}}(t)$, is not related to the semigroup generated by $\mathcal{A}$, say $T(t)$, e.g. a relation like $T_{\mathcal{L}}(t) = \mathcal{L}^{-1} T(t) \mathcal{L}$ *does not* hold. In fact, as we shall see in Chapter 4, the eigenvalues of $\mathcal{A}$ and $\mathcal{A}_{\mathcal{L}}$ are completely different.

Let us stress that the evaluation on the boundary used for the port-variables in the theorems above and below is done on $\mathcal{L}\, x$ rather than only on x, i.e., see (2.10),

$$\begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} = R_{\text{ext}}\, \tau(\mathcal{L}\, x).$$

Next we define boundary control systems for the class of systems (2.1).

Theorem 2.14: *Let W be a full rank real matrix of size $nN \times 2nN$ and consider the operator $\mathcal{J}\mathcal{L}$ as described above. If W satisfies $W \Sigma W^T \geq 0$, where Σ is defined in* (2.9)*, then the following system*

$$\frac{\partial x}{\partial t}(t) = \mathcal{J}\mathcal{L}x(t), \quad \textit{or equivalently} \quad \Big(\dot{x}(t),\, f_{\partial,\mathcal{L}x}(t),\, \mathcal{L}x(t),\, e_{\partial,\mathcal{L}x}(t) \Big) \in \mathcal{D}_{\mathcal{J}}$$

defined on the state space X (see (2.33)*) with input*

$$u(t) = \mathfrak{B}\, x(t) = W \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}$$

is a boundary control system on X. Furthermore, the operator $\mathcal{A}_{\mathcal{L}} = \mathcal{J}\mathcal{L}$ with domain

$$D(\mathcal{A}_{\mathcal{L}}) = \left\{ x \in X \mid \mathcal{L}x \in H^N(a,b;\mathbb{R}^n), \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix} \in \ker W \right\}, \tag{2.40}$$

generates a contraction semigroup on X.

Let $\widetilde{W}$ be a full rank matrix of size $nN \times 2nN$ such that $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ is invertible. If we define the linear mapping $\mathcal{C} : D(\mathcal{J}\mathcal{L}) \to \mathbb{R}^{nN}$ (where $D(\mathcal{J}\mathcal{L})$ is given in (2.36)) as,

$$\mathcal{C}x(t) := \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix} \tag{2.41}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{2.42}$$

then for $u \in C^2(0,\infty;\mathbb{R}^{nN})$, $\mathcal{L}x(0) \in H^N(a,b;\mathbb{R}^n)$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\frac{d}{dt}H(t) = \frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = \frac{1}{2}\begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{2.43}$$

where

$$P^{-1}_{W,\tilde{W}} = \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix} \Sigma \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^T = \begin{bmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{bmatrix}. \tag{2.44}$$

Furthermore, the invertibility of the matrix $\left(\begin{smallmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{smallmatrix}\right)$ is equivalent to the invertibility of $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$.

PROOF: We give a sketch of the proof, for details see [LZM05, §4] or the proof of Theorem 2.10. That we have a boundary control system in the sense of Definition 1.10 follows from Theorem 2.13 and Theorem 2.6 (since W has full row-rank). The balance equation (2.43) follows by noticing that (the differentiability follows by using the same ideas as in the proof of Theorem 2.10)

$$\begin{aligned} \frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} &= \frac{d}{dt}\langle x(t), x(t)\rangle_{\mathcal{L}} = \langle \dot{x}(t), x(t)\rangle_{\mathcal{L}} + \langle x(t), \dot{x}(t)\rangle_{\mathcal{L}} \\ &= \langle \mathcal{J}\mathcal{L}x(t), x(t)\rangle_{\mathcal{L}} + \langle x(t), \mathcal{J}\mathcal{L}x(t)\rangle_{\mathcal{L}} \\ &= \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}^T \Sigma \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}, \end{aligned} \tag{2.45}$$

and that the relation between the port-variables and the input-output is given by

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix} \begin{bmatrix} f_{\partial,x} \\ e_{\partial,x} \end{bmatrix},$$

where $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ is nonsingular. See [LZM05, §4] or [LZM04] for more details. ■

Remark 2.15. Theorem 2.14 together with equation (2.45) implies that the semigroup generator $\mathcal{A}_{\mathcal{L}}$ satisfies (see (2.39))

$$\langle \mathcal{A}_{\mathcal{L}} x, x \rangle_{\mathcal{L}} = \frac{1}{2} \begin{pmatrix} 0 & y^T \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} 0 \\ y \end{pmatrix} \quad \forall x \in D(\mathcal{A}_{\mathcal{L}}).$$

♣

It is easy to see that the system just defined is a dissipative system as described in Section 1.8. Here the matrix $P_{W,\tilde{W}}$ determines the supply rate of the system and since it only depends on the matrices W and $\widetilde{W}$ we can select a desired supply rate. The two theorems below described how we can obtain, in particular, impedance passive and scattering passive systems.

Theorem 2.16 (Impedance energy preserving): *Consider the boundary control system described in Theorem 2.14 and let W and $\widetilde{W}$ be $nN \times 2nN$ matrices with W satisfying* (2.20) *and $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ nonsingular. Then for $u \in C^2(0,\infty;\mathbb{R}^{nN})$, $\mathcal{L}x(0) \in H^N(a,b;\mathbb{R}^n)$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied*

$$\frac{d}{dt} H(t) = \frac{1}{2} \frac{d}{dt} \|x(t)\|^2_{\mathcal{L}} = u(t)^T y(t)$$

if and only if the following conditions are satisfied

$$W = S \begin{bmatrix} I + V, & I - V \end{bmatrix}, \quad \textit{with } S \textit{ nonsingular and } V \textit{ unitary,} \tag{2.46}$$

$$\widetilde{W} = \widetilde{S} \begin{bmatrix} I + \widetilde{V}, & I - \widetilde{V} \end{bmatrix}, \quad \textit{with } \widetilde{S} \textit{ nonsingular and } \widetilde{V} \textit{ unitary, and} \tag{2.47}$$

$$I = 2\widetilde{S}(I - \widetilde{V} V^T) S^T. \tag{2.48}$$

As a consequence these matrices satisfy

$$W^T \widetilde{W} + \widetilde{W}^T W = \Sigma, \tag{2.49}$$

where $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ is defined in (2.9)*. Furthermore, under the condition* (2.46) *the associated semigroup is unitary and $\mathcal{A}^*_{\mathcal{L}} = -\mathcal{A}_{\mathcal{L}}$ and $D(\mathcal{A}_{\mathcal{L}}) = D(\mathcal{A}^*_{\mathcal{L}})$.*

PROOF: From Theorem 2.14 we see that we only have to check that $P_{W,\tilde{W}}$ equals $\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$. By (2.44) this is equivalent to $W \Sigma W^T = \widetilde{W} \Sigma \widetilde{W}^T = 0$ and $\widetilde{W} \Sigma W^T = I$. This follows easily by using the expressions (2.46) and (2.47) for W and $\widetilde{W}$ together with the condition (2.48).

Since $P_{W,\tilde{W}} = \left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]^{-T} \Sigma \left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]^{-1}$ equals $\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ we must have

$$\Sigma = \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^T \Sigma \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix} = \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^T \begin{bmatrix} \widetilde{W} \\ W \end{bmatrix}, \tag{2.50}$$

from which (2.49) follows. ■

Theorem 2.17 (Scattering energy preserving): *Consider the boundary control system described in Theorem 2.14 and let W and $\widetilde{W}$ be $nN \times 2nN$ matrices with W satisfying* (2.20) *and $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ nonsingular. Then for $u \in C^2(0,\infty;\mathbb{R}^{nN})$, $\mathcal{L}x(0) \in H^N(a,b;\mathbb{R}^n)$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied*

$$\frac{d}{dt}H(t) = \frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = \|u(t)\|^2_{\mathbb{R}^{nN}} - \|y(t)\|^2_{\mathbb{R}^{nN}}$$

if and only if the following conditions are satisfied

$$\begin{aligned} W &= S\begin{bmatrix} I+V, & I-V \end{bmatrix}, \quad \text{with } 4S(I-VV^T)S^T = I, \\ \widetilde{W} &= \widetilde{S}\begin{bmatrix} -(I+V^T), & I-V^T \end{bmatrix}, \quad \text{with } 4\widetilde{S}(I-V^TV)\widetilde{S}^T = I. \end{aligned} \tag{2.51}$$

As a consequence these matrices satisfy

$$2W^TW - 2\widetilde{W}^T\widetilde{W} = \Sigma, \tag{2.52}$$

where $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ is defined in (2.9).

PROOF: In this case it is easy to check that $W\Sigma W^T = \frac{1}{2}I$, $\widetilde{W}\Sigma\widetilde{W}^T = -\frac{1}{2}I$ and $W\Sigma\widetilde{W}^T = 0$. Hence from (2.44) we obtain $P_{W,\tilde{W}}$ equals $\left[\begin{smallmatrix} 2I & 0 \\ 0 & -2I \end{smallmatrix}\right]$. The result now follows from Theorem 2.14. ■

Remark 2.18. Once $W = \begin{bmatrix} W_1, & W_2 \end{bmatrix}$ is chosen the matrices S and V can be found easily by

$$S = \frac{1}{2}(W_1 + W_2), \quad V = (W_1 + W_2)^{-1}(W_1 - W_2).$$

See Lemma A.1 of [LZM05] for details. ♣

Example 2.19 (Timoshenko beam) Consider the Timoshenko type beam equations described in Example 1.2. This model can be written as a system (2.1) by selecting the state variables

$$\begin{aligned} x_1 &= \tfrac{\partial w}{\partial z} - \phi : && \text{shear displacement,} \\ x_2 &= \rho \tfrac{\partial w}{\partial t} : && \text{transverse momentum distribution,} \\ x_3 &= \tfrac{\partial \phi}{\partial z} : && \text{angular displacement,} \\ x_4 &= I_\rho \tfrac{\partial \phi}{\partial t} : && \text{angular momentum distribution.} \end{aligned}$$

Then the model of the beam can be rewritten as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P_1} \frac{\partial}{\partial z}\begin{bmatrix} K\,x_1 \\ \frac{1}{\rho}\,x_2 \\ EI\,x_3 \\ \frac{1}{I_\rho}\,x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{P_0} \begin{bmatrix} K\,x_1 \\ \frac{1}{\rho}\,x_2 \\ EI\,x_3 \\ \frac{1}{I_\rho}\,x_4 \end{bmatrix}. \tag{2.53}$$

From here we can see that the operator $\mathcal{J}$ is a first order differential operator of the form (2.2)–(2.3). It thus follows that $n = 4$, $\mathcal{L} = \operatorname{diag}\{K,\, \frac{1}{\rho},\, EI,\, \frac{1}{I_\rho}\} > 0$, and P_1 is nonsingular. The energy of the system is known to be

$$\begin{aligned} E &= \frac{1}{2}\int_a^b \left(K|x_1|^2 + \frac{1}{\rho}|x_2|^2 + EI|x_3|^2 + \frac{1}{I_\rho}|x_4|^2 \right) dz \\ &= \frac{1}{2}\int_a^b x^T(z)(\mathcal{L}x)(z)\,dz = \frac{1}{2}\,\|x\|_{\mathcal{L}}^2, \end{aligned} \tag{2.54}$$

where $x = [x_1,\, x_2,\, x_3,\, x_4]^T$. Note that the energy function under this selection of state variables does not include derivatives of the energy variables. Hence we do not need to include derivatives in the definition of the norm of the state space. This also motivates the selection of the state space (2.33). Next we find the port-variables which are given by

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}\begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} (\rho^{-1}x_2)(b)-(\rho^{-1}x_2)(a) \\ (Kx_1)(b)-(Kx_1)(a) \\ (I_\rho^{-1}x_4)(b)-(I_\rho^{-1}x_4)(a) \\ (EIx_3)(b)-(EIx_3)(a) \\ (Kx_1)(b)+(Kx_1)(a) \\ (\rho^{-1}x_2)(b)+(\rho^{-1}x_2)(a) \\ (EIx_3)(b)+(EIx_3)(a) \\ (I_\rho^{-1}x_4)(b)+(I_\rho^{-1}x_4)(a) \end{bmatrix}. \tag{2.55}$$

Assume that the beam is clamped at both sides. This corresponds to the following boundary conditions (inputs)

$$\frac{1}{\rho(a)}\,x_2(a) = \frac{1}{I_\rho(a)}\,x_4(a) = \frac{1}{\rho(b)}\,x_2(b) = \frac{1}{I_\rho(b)}\,x_4(b) = 0. \tag{2.56}$$

Note that this corresponds to setting the velocities at the ends of the beam to zero. From the equation above we can see that W can be selected as

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad W\Sigma W^T = 0.$$

Since W satisfies the conditions on Theorem 2.14 we have that under this boundary conditions the system (2.53) is a boundary control system. As output we can choose

$$y = \begin{bmatrix} -K(a)x_1(a) \\ -(EI)(a)x_3(a) \\ K(b)x_1(b) \\ (EI)(b)x_3(b) \end{bmatrix}, \quad \text{with} \quad \widetilde{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then, from equation (2.30) we obtain again that

$$P_{W,\tilde{W}}^{-1} = P_{W,\tilde{W}} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

which, in turn gives from (2.29) that

$$\frac{d}{dt}E(t) = \frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \frac{1}{2}\begin{bmatrix} u^T(t) & y^T(t) \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = u^T(t)\, y(t).$$

Observe that if we let $u(t) = 0$ for $t \geq 0$, then there is no change in the energy of the system. $*$

2.4. Relation with the characteristic curves

Characteristics curves can be used in the analysis of systems described by PDEs. In some cases they can be used to find a solution of a given PDE, see [Zau89, Ch.2 and 3] and [Col04, Ch. 5]. They also help to understand how the Cauchy data imposed for a given PDE (linear or nonlinear) determines a solution of that PDE, see [Joh78]. They can even be used for control and disturbance rejection in BCS, see [BCANM05]. Loosely speaking, the characteristic curves are a family of curves on which the solution of a given PDE remain constant. In this section we study the case $N = 1$ and show that the eigenvalues of the matrix P_1 determines the characteristic curves of the PDE (2.1a). First we introduce some notation and then we define the characteristic curves.

The notation of multi-indices (Schwartz notation) is very convenient when dealing with PDE's. A multi-index is a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ whose components are non-negative integers. The notation $\alpha \geq \beta$ indicates $\alpha_i \geq \beta_i$ for each i.

We also have

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!;$$

moreover, for any vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set

$$\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}. \tag{2.57}$$

The following notation for partial derivatives is very common:

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

In this notation the general m-th order linear differential equation for a function $u(x_1, \ldots, x_n)$ takes the form

$$L(\mathbf{x}, D)u = \sum_{|\alpha| \leq m} A_\alpha(\mathbf{x}) D^\alpha u = B(\mathbf{x}). \tag{2.58}$$

The same expression describes the general m-th order system of differential equations, in this case we interpret u and B as column vectors and A_α as square matrices.

Definition 2.20. The *symbol* of the expression $L(\mathbf{x}, D)$ as given above is (with the notation (2.57))

$$L(\mathbf{x}, i\xi) := \sum_{|\alpha| \leq m} A_\alpha(\mathbf{x})(i\xi)^\alpha. \tag{2.59}$$

The *principal part* of the symbol is

$$L_p(\mathbf{x}, i\xi) := \sum_{|\alpha| = m} A_\alpha(\mathbf{x})(i\xi)^\alpha. \tag{2.60}$$

♣

In the case of constant parameters, note that the dependance of L and L_p on $\mathbf{x}$, can be dropped, i.e., $L(i\xi)$.

Example 2.21 The symbol of Laplace's operator $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is $-\xi_1^2 - \xi_2^2$, the symbol of the heat operator $\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2}$ is $i\xi_1 + \xi_2^2$, and the symbol of the wave operator $\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ is $-\xi_1^2 + \xi_2^2$. For the Laplace and wave operator, the symbols are equal to their principal parts; the principal part for the heat operator is ξ_2^2. For the equation

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial z}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P_1 \frac{\partial}{\partial z}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.61}$$

we have that the principal part and the symbol are given by $i\xi_1 I - i\xi_2 P_1 = i\begin{bmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_1 \end{bmatrix}$.

∗

Definition 2.22. Let L be the m-th order differential operator defined in (2.58). The surface $\phi(x_1, \dots, x_n)$ is a *characteristic surface* at a point $\mathbf{x}$ if $\phi(\mathbf{x}) = 0$ and, in addition,

$$\det L_p(\mathbf{x}, \nabla\phi) = 0. \tag{2.62}$$

A surface is called characteristic if it is characteristic at each of its points. ♣

Example 2.23 Consider the surfaces $\phi_1(t, z) = z - c$ and $\phi_2(t, z) = z - t - c$, where $c \in \mathbb{R}$, and the PDE (2.61). Clearly, the surface $\phi_1 = 0$ is noncharacteristic since $\nabla\phi_1 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $\det L_p(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \nabla\phi_1) = \det \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \neq 0$. On the other hand, the surface $\phi_2 = 0$ is characteristic since $\nabla\phi_2 = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ and $\det L_p(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \nabla\phi_2) = \det \left[\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right] = 0$.

Next consider the heat equation $\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial z^2} = 0$. We have that the surfaces $\phi(t, z) = t - c$ are characteristic since $\nabla\phi = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $L_p(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \nabla\phi) = 0$. Note that the surfaces $\phi_2(t, z) = z - c$ are noncharacteristic since $\nabla\phi_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $L_p(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \nabla\phi) = 1$. ∗

Now we can describe how the characteristic curves are related to the PDE (2.1a). As we mention earlier, we study the case $N = 1$ since in this case the characteristics are more meaningful. Note that in the case $N = 1$ the principal symbol of the PDE (2.1a) is $L_p\left(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right]\right) = \xi_1 I - P_1\mathcal{L}(z)\,\xi_2$ since

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial(\mathcal{L}x)}{\partial z} + P_0\mathcal{L}\,x \quad \Rightarrow \quad \frac{\partial x}{\partial t} - P_1\mathcal{L}\frac{\partial x}{\partial z} - (P_1\mathcal{L}' + P_0\mathcal{L})\,x = 0.$$

It thus follows from Definition 2.22 that the characteristic surfaces are determined by the functions $\phi = 0$ which satisfy

$$\det L_p\left(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \nabla\phi\right) = \det\left(\frac{\partial\phi}{\partial t} I - P_1\mathcal{L}(z)\frac{\partial\phi}{\partial z}\right) = 0.$$

Clearly the functions $\phi_i = \lambda_i t + z + k$ (or equivalently, $\phi_i = q_i t + p_i z + k$ where $\lambda_i = q_i/p_i$), with $k \in \mathbb{R}$ and $\lambda_i(z)$, $i = 1, \dots, n$, being an eigenvalue of $P_1\mathcal{L}(z)$, are the characteristic surfaces. This follows since $\nabla\phi_i = \left[\begin{smallmatrix} \lambda_i \\ 1 \end{smallmatrix}\right]$ and $\det L_p\left(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \nabla\phi_i\right) = \det(\lambda_i I - P_1\mathcal{L}) = 0$, see Definition 2.22. Recall that $\mathcal{L}$ is assumed to be continuous and coercive, see Theorem 2.14, and P_1 is a constant symmetric nonsingular matrix. Hence, the eigenvalues of $P_1\mathcal{L}$ are continuous functions of z and, moreover, we must have that either $\lambda_i > 0$ or $\lambda_i < 0$, for $i = \{1, 2, \dots, n\}$. In fact, since $\mathcal{L}$ is coercive, the number of positive and negative eigenvalues of $P_1\mathcal{L}$ is the same as those of P_1. We can thus conclude, see [Zau89, Ch.2 and 3] and [Col04, Ch. 5], that if the eigenvalue λ_i satisfies $\lambda_i < 0$, then the characteristics move in the direction shown in Figure 2.1, i.e., the eigenvalue determines the slope of the characteristic curves. The main point of finding the characteristic curves is that the solution remains constant along this lines, see [Zau89] or [Col04]. Hence, if we know the solution of a PDE along a noncharacteristic surface, then we can determine the solution in the whole (z, t)-plane (this is one of the ideas in the proof of Holmgren's theorem, see [Joh78]). *From all this we can conclude that the number of negative eigenvalues of P_1 (since $\mathcal{L}$ is coercive) determine the number of boundary conditions imposed on $z = a$ and the number of positive eigenvalues determine the number of boundary conditions on $z = b$.*

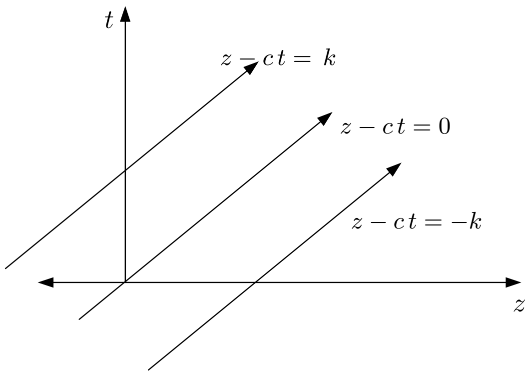

Figure 2.1.: Characteristic curves in the (z, t)-plane.

2.5. Properties of the semigroup generator

The validity of the conditions in Theorem 2.13 for the operator $\mathcal{A}_{\mathcal{L}}$ guarantees the existence of a unique solution for the PDE (2.1) when the input u is set to zero. Later we show that a deeper study of this operator can lead to the establishment of other attributes of the system (2.1) such as stability and controllability, see [CZ95b] for more details on the general theory around this. In this section we study some properties of the operator $\mathcal{A}_{\mathcal{L}}$ which are useful to prove results presented in the later chapters.

2.5.1. Adjoint operator

Here we present the expression that determines the adjoint operator of $\mathcal{A}_{\mathcal{L}}$, i.e., $\mathcal{A}^*_{\mathcal{L}}$. We represent this operator in terms of the matrices S and V that determines W, see (2.20). The proof of this result is based on equation (2.21). Recall that the matrix W satisfies $W\Sigma W^T \geq 0$ if and only if it can be parameterized as equation (2.20). First we give the definition of the adjoint of an unbounded operator.

The adjoint operator is defined by

$$\mathcal{A}_{\mathcal{L}}^{*}u = \{w \in L_2(a,b)^n \mid \forall\, y \in D(\mathcal{A}_{\mathcal{L}}) \text{ we have } \langle \mathcal{A}_{\mathcal{L}} y, u\rangle_{\mathcal{L}} = \langle y, w\rangle_{\mathcal{L}}\}\,, \tag{2.63a}$$

with domain

$$u \in D(\mathcal{A}_{\mathcal{L}}^{*}) \iff \exists\, w \in L_2(a,b)^n \text{ s.t. } \forall\, y \in D(\mathcal{A}_{\mathcal{L}}) \text{ we have } \langle \mathcal{A}_{\mathcal{L}} y, u\rangle_{\mathcal{L}} = \langle y, w\rangle_{\mathcal{L}}\,. \tag{2.63b}$$

Using this we proceed to find $\mathcal{A}_{\mathcal{L}}^{*}$.

Theorem 2.24: *Let $W = S\begin{bmatrix} I+V, & I-V \end{bmatrix}$ where $V \in \mathbb{R}^{nN\times nN}$ satisfies $VV^T \leq I$ and $S \in \mathbb{R}^{nN\times nN}$ is nonsingular. Consider the operator $\mathcal{A}_{\mathcal{L}}$ as described in Theorem* 2.13 *on the state space X given by* (2.33). *Then the adjoint of $\mathcal{A}_{\mathcal{L}}$ is given by*

$$\mathcal{A}_{\mathcal{L}}^{*}\, x = -\mathcal{J}\mathcal{L}\, x \tag{2.64}$$

and

$$D(\mathcal{A}_{\mathcal{L}}^{*}) = \left\{ x \in X \;\middle|\; \begin{matrix} \mathcal{L}x \in H^N(a,b;\mathbb{R}^n), \\ \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} \in \ker \begin{bmatrix} -(I+V^T), & I-V^T \end{bmatrix} \end{matrix} \right\}. \tag{2.65}$$

PROOF: The proof follows the same lines as the proof of Lemma 4.1 of [LZM04]. By noticing that every function on $H^N(a,b)^n$, say $\mathcal{L}y$, which is zero at the boundary is in the domain of $\mathcal{A}_{\mathcal{L}}$ and by using (2.63), i.e., $\langle \mathcal{A}_{\mathcal{L}} y, u\rangle_{\mathcal{L}} = \langle \mathcal{J}\mathcal{L}y, \mathcal{L}u\rangle = \langle y, w\rangle_{\mathcal{L}} = \langle \mathcal{L}y, w\rangle$, it is easy to show that every $\mathcal{L}u \in D(\mathcal{A}_{\mathcal{L}}^{*})$ must be an element of $H^N(a,b)^n$. Thus we can use (2.35), from which we can write for all $x_1 \in D(\mathcal{A}_{\mathcal{L}})$ and $x_2 \in D(\mathcal{A}_{\mathcal{L}}^{*})$

$$\langle \mathcal{A}_{\mathcal{L}} x_1, x_2\rangle_{\mathcal{L}} = \langle \mathcal{J}\mathcal{L}x_1, x_2\rangle_{\mathcal{L}} = -\langle x_1, \mathcal{J}\mathcal{L}x_2\rangle_{\mathcal{L}} + \begin{bmatrix} f_{\partial,\mathcal{L}x_1} \\ e_{\partial,\mathcal{L}x_1} \end{bmatrix}^T \Sigma \begin{bmatrix} f_{\partial,\mathcal{L}x_2} \\ e_{\partial,\mathcal{L}x_2} \end{bmatrix},$$

where Σ is given in (2.9). Since $\left[\begin{smallmatrix} f_{\partial,\mathcal{L}x_1} \\ e_{\partial,\mathcal{L}x_1} \end{smallmatrix}\right]$ lies in the kernel of W we get, from equation (2.21), that $\left[\begin{smallmatrix} f_{\partial,\mathcal{L}x_1} \\ e_{\partial,\mathcal{L}x_1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I-V \\ -(I+V) \end{smallmatrix}\right] l$ for some $l \in \mathbb{R}^{nN}$. Hence

$$\begin{aligned} \langle \mathcal{A}_{\mathcal{L}} x_1, x_2\rangle_{\mathcal{L}} &= \langle x_1, -\mathcal{J}\mathcal{L}x_2\rangle_{\mathcal{L}} + l^T \begin{bmatrix} I-V^T, & -(I+V^T) \end{bmatrix} \Sigma \begin{bmatrix} f_{\partial,\mathcal{L}x_2} \\ e_{\partial,\mathcal{L}x_2} \end{bmatrix} \\ &= \langle x_1, -\mathcal{J}\mathcal{L}x_2\rangle_{\mathcal{L}} + l^T \begin{bmatrix} -(I+V^T), & I-V^T \end{bmatrix} \begin{bmatrix} f_{\partial,\mathcal{L}x_2} \\ e_{\partial,\mathcal{L}x_2} \end{bmatrix}. \end{aligned}$$

Using the defining condition (2.63) and the fact that the equality above must hold for all $l \in \mathbb{R}^{nN}$, we conclude that

$$\begin{bmatrix} f_{\partial,\mathcal{L}x_2} \\ e_{\partial,\mathcal{L}x_2} \end{bmatrix} \in \ker \begin{bmatrix} -(I+V^T), & I-V^T \end{bmatrix} \quad \text{and} \quad \mathcal{A}_{\mathcal{L}}^{*}\, x_2 = -\mathcal{J}\mathcal{L}\, x_2.$$

This concludes the proof. ■

2.5.2. Spectrum and compactness of the resolvent operator

In this section we study the spectrum and the resolvent operator of $\mathcal{A}_{\mathcal{L}}$. First we look at the eigenvalues and then we show that the resolvent operator is compact, which gives that the spectrum is comprised only of eigenvalues and the only accumulation point is at infinity.

As mentioned, we begin with the study of the eigenvalues of $\mathcal{A}_{\mathcal{L}}$. Since $\mathcal{A}_{\mathcal{L}}$ generates a contraction semigroup we know that $\mathrm{Re}\,\langle \mathcal{A}_{\mathcal{L}}x, x\rangle_{\mathcal{L}} \leq 0$ for all $x \in D(\mathcal{A}_{\mathcal{L}})$. Let x_1 be any eigenvector of $\mathcal{A}_{\mathcal{L}}$ with corresponding eigenvalue λ. Then we have that

$$0 \geq \mathrm{Re}\,\langle \mathcal{A}_{\mathcal{L}}x_1, x_1\rangle_{\mathcal{L}} = \mathrm{Re}\,\langle \mathcal{A}_{\mathcal{L}}x_1, \mathcal{L}x_1\rangle = \mathrm{Re}\,\langle \lambda x_1, \mathcal{L}x_1\rangle = \mathrm{Re}\,\lambda\,\langle x_1, \mathcal{L}x_1\rangle\,.$$

This implies (by the coercivity of $\mathcal{L}$) that all eigenvalues of $\mathcal{A}_{\mathcal{L}}$ satisfy $\mathrm{Re}\,\lambda \leq 0$.

To prove the compactness of the resolvent operator we will make use of the following theorem, which is taken from [NS00, Theorem 7.6.4]. In this theorem, a Banach space X_i with norm $\|\cdot\|_i$ is denoted by $(X_i, \|\cdot\|_i)$.

Theorem 2.25: *Let $(X_1, \|\cdot\|_1)$ be compact in $(X_2, \|\cdot\|_2)$, where $(X_2, \|\cdot\|_2)$ is a Banach space and let T be a linear operator on X_2. Assume that the domain $D(T)$ lies in X_1. If T satisfies $\|Tx\|_2 \geq a\,\|x\|_1$ for some $a > 0$ and all $x \in D(T)$, then T^{-1} exist and is a compact operator. Furthermore, there are constants A, B such that*

$$\left\|T^{-1}y\right\|_1 \leq A\,\|y\|_2\,, \qquad \left\|T^{-1}y\right\|_2 \leq B\,\|y\|_2\,,$$

for all $y \in D(T)$.

First we prove the result for $\mathcal{L} = I$ and then we extend the result including the operator $\mathcal{L}$. So, we first assume that $\mathcal{L} = I$. In this case the semigroup generator is described in Theorem 2.8. From this, we can see that the domain of $\mathcal{A}_{\mathcal{L}} = \mathcal{A}$ lies in $H^N(a,b)^n$. The main property that is used to prove the compactness of the resolvent operator is that the imbedding of $H^N(a,b)$ into $L_2(a,b) = H^0(a,b)$ is compact.

Theorem 2.26: *Consider the operator $\mathcal{A}$ as described in Theorem 2.8 on the state space $L_2(a,b;\mathbb{R}^n)$. If W satisfies $W\Sigma W^T \geq 0$, then the resolvent operator $(\lambda - \mathcal{A})^{-1}$, for $\lambda \in \rho(\mathcal{A})$, is a compact operator. As a consequence, the spectrum of $\mathcal{A}$, $\sigma(\mathcal{A})$, consists only of isolated eigenvalues with finite multiplicity. That is, $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$.*

PROOF: In the notation of Theorem 2.25 we clearly have, in this case, that $X_1 = H^N(a,b)^n$ and $X_2 = L_2(a,b)^n$. That $H^N(a,b)^n$ is compact in $L_2(a,b)^n$ follows from Theorem 2 and Remark 5 on pages 124 and 117, respectively, of [DL85a]; see also [Aub00, Proposition 7.5.3].

We want to prove that the resolvent of $\mathcal{A}$ is compact for all $\lambda > 0$, and hence for all $\lambda \in \rho(\mathcal{A})$. Since $\mathcal{A}$ generates a contraction semigroup, we know that $\lambda > 0$ is in the resolvent set of $\mathcal{A}$. Thus, $T = (\lambda - \mathcal{A})$ is boundedly invertible and hence it satisfies $\|Tx\|_{L_2(a,b)} \geq a\,\|x\|_{H^N(a,b)}$ for all $x \in D(\mathcal{A})$, see [NS00, §7.6]. Thus, the result follows from Theorem 2.25. The assertion on the spectrum of $\mathcal{A}$ follows from [Kat95, Th.6.29, ch3]. ■

The extension to the operator $\mathcal{A}_\mathcal{L}$ follows easily from the following lemma.

Lemma 2.27: *Let X be a Hilbert space and A and L be operators on X. Furthermore, assume that $L \in \mathcal{L}(X)$ is boundedly invertible and that A has compact resolvent. If $(I - AL)$ with domain $\{x \in X \mid Lx \in D(A)\}$ is boundedly invertible, then $(I - AL)^{-1}$ is compact.* ♡

PROOF: We must prove that

$$(I - AL)^{-1} x_n = y_n \tag{2.66}$$

has a converging subsequence for every bounded sequence x_n, see Theorem 8.1-3 of [Kre89]. Let $\alpha \in \rho(A)$. Then

$$\begin{aligned} x_n &= (I - AL)y_n = (I - \alpha L + \alpha L - AL)y_n \\ &= (\alpha I - A)Ly_n + (I - \alpha L)y_n. \end{aligned}$$

Or equivalently,

$$Ly_n = (\alpha I - A)^{-1}[x_n - (I - \alpha L)y_n].$$

Since $(I - AL)$ is boundedly invertible, we have that $\{y_n\}$ is bounded, see (2.66). Hence $x_n - (I - \alpha L)y_n$ is a bounded sequence. Since $(\alpha I - A)^{-1}$ is compact (by assumption), $\{Ly_n\}$ has a convergent subsequence. Using the fact that L is boundedly invertible, gives that $\{y_n\}$ has a convergent subsequence. ■

Recall that $\mathcal{A}_\mathcal{L}$ generates a contraction semigroup and hence $I - \mathcal{A}_\mathcal{L}$ is boundedly invertible. Hence, combining the lemma above with Theorem 2.26 gives the following result.

Theorem 2.28: *Consider the operator $\mathcal{A}_\mathcal{L}$ as described in Theorem 2.13 on the state space X given by (2.33). If W satisfies $W\Sigma W^T \geq 0$, then the resolvent operator $(\lambda - \mathcal{A}_\mathcal{L})^{-1}$, for $\lambda \in \rho(\mathcal{A}_\mathcal{L})$, is a compact operator. As a consequence, the spectrum of $\mathcal{A}_\mathcal{L}$, $\sigma(\mathcal{A}_\mathcal{L})$, consists only of isolated eigenvalues with finite multiplicity. That is, $\sigma(\mathcal{A}_\mathcal{L}) = \sigma_p(\mathcal{A}_\mathcal{L})$.*

2.6. System nodes and boundary control systems

In this section we study the relation between the system node and boundary control systems and we describe how to write the class of boundary control systems (2.1) as a system node.

The system node is another possibility to represent infinite-dimensional systems and several results for systems described by the system node are available in the literature. See for instance [Sta05], [Sta02], [MSW03] and the references therein. Thus, writing a boundary control system as a system node allows us to apply the available results for system nodes to the BCS. We begin by giving a brief description of system nodes.

2.6.1. System nodes

Many finite- and infinite-dimensional linear systems can be described by the equations

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t), \quad t \geq 0, \\ x(0) &= x_0 \end{aligned} \tag{2.67}$$

where $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$ with the input space U, the state space X and the output space Y, being Hilbert spaces. The operator A is generally the generator of a C_0-semigroup. Here the operators B and C are not necessarily bounded.

The system node (see [Sta05], [MSW03, §2]) has been introduced as a generalization of this set of equations. The system node can be thought of as the block operator $\mathcal{S} = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ from $X \times U$ to $X \times Y$, which allows to rewrite equation (2.67) as follows

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \mathcal{S} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0.$$

However, it is not obvious how to represent boundary control systems of the form (2.1) either as the set of equations (2.67) or as a system node. This section describes how to formulate boundary control system (BCS) as a system node. First we introduce some functional spaces which are needed in the rest of this section.

Proposition 2.29 (Proposition 2.1, [MSW03]): *Let X be a Hilbert space and let $A : D(A) \subset X \rightarrow X$ be a closed, densely defined linear operator with a nonempty resolvent set $\rho(A)$. Take $\alpha \in \rho(A)$.*

(i) *Let* $X_1 = D(A)$ *and define* $\|x\|_{X_1} = \|(\alpha - A)x\|_X$. *Then* $\|\cdot\|_{X_1}$ *is a norm on* X_1 *which makes* X_1 *into a Hilbert space, and* $A \in \mathcal{L}(X_1; X)$. *The operator* $(\alpha - A)^{-1}$ *maps* X *isometrically onto* X_1.

(ii) *Let* X_{-1} *be the completion of the space* X *with respect to the norm* $\|x\|_{X_{-1}} = \left\|(\alpha - A)^{-1}x\right\|_X$. *Then* X *is continuously and densely embedded in* X_{-1}, *and* A *has a unique extension to an operator* A_e *in* $\mathcal{L}(X; X_{-1})$. *The operator* $(\alpha - A_e)^{-1}$ *maps* X_{-1} *isometrically onto* X. *Moreover,* A_e *and* A *are unitarily similar:* $A_e = (\alpha - A_e)A(\alpha - A_e)^{-1}$.

(iii) *If* A *is the generator of a* C_0*-semigroup* $\mathbb{T}(t)$ *on* X, *then the restriction* $\mathbb{T}_1(t) = \mathbb{T}(t)_{|X_1}$ *of* $\mathbb{T}(t)$ *to* X_1 *is a* C_0*-semigroup on* X_1. *The semigroup* $\mathbb{T}(t)$ *has a unique extension to a* C_0*-semigroup* $\mathbb{T}_e(t)$ *on* X_{-1} *which is unitarily similar to* $\mathbb{T}(t)$, *since* $\mathbb{T}_e(t) = (\alpha - A_e)\mathbb{T}(t)(\alpha - A_e)^{-1}$. ♡

Notice that $X_1 \subset X \subset X_{-1}$ with continuous and dense embedding. Often X_{-1} is defined in an equivalent form as the dual of $D(A^*)$. Dual version of the spaces X_1 and X_{-1} can also be constructed by replacing A with the adjoint, A^*, of A. The resulting spaces are denoted by X_1^d (the equivalent of X_1) and by X_{-1}^d (the equivalent of X_{-1}) respectively. It can be checked that X_{-1}^d is the dual of X_1 with respect to the pivot space X. Likewise, X_1^d is the dual of X_{-1}. Thus, $A_e^* \in \mathcal{L}(X; X_{-1}^d)$ can be interpreted as the (bounded) adjoint of the operator $A \in \mathcal{L}(X_1; X)$.

Now it is possible to define the system node as follows

Definition 2.30 (Malinen et al. [MSW03]). Let U, X and Y be Hilbert spaces. An operator

$$\mathcal{S} := \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : D(\mathcal{S}) \to \begin{bmatrix} X \\ Y \end{bmatrix}$$

is called an *operator node* on (U, X, Y) if it has the following structure:

(i) The operator A defined by $Ax = A\&B \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ on $D(A) = \{x \in X \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in D(\mathcal{S})\}$ is a densely defined operator on X with a nonempty resolvent set (which we extend to an operator $A_e \in \mathcal{L}(X; X_1)$ as explained in Proposition 2.29);

(ii) $B \in \mathcal{L}(U; X_{-1})$;

(iii) $D(\mathcal{S}) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \middle| \; A_e x + Bu \in X \right\}$, and $A\&B = [A_e, \; B]_{|D(\mathcal{S})}$;

(iv) $C\&D \in \mathcal{L}(D(\mathcal{S}); Y)$ with respect to the graph norm of $A\&B$ (with values in X):

$$\left\| \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{D(\mathcal{S})}^2 := \|x\|_X^2 + \|u\|_U^2 + \|A_e x + Bu\|_X^2 . \tag{2.68}$$

If in addition to the above, A generates a C_0-semigroup on X, then $\mathcal{S}$ is called a *system node*. ♣

Here we call $A \in \mathcal{L}(X_1, X)$ the *main operator* of the node, $B \in \mathcal{L}(U, X_{-1})$ is its *control operator*, and $C\&D \in \mathcal{L}(D(\mathcal{S}), Y)$ is its *combined observation/feedthrough operator*. From the last operator we can extract $C \in \mathcal{L}(X_1, Y)$, the *observation operator* of $\mathcal{S}$, defined by

$$Cx := C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad x \in X_1. \tag{2.69}$$

From the boundedness of the operators A_e and B combined with the characterization of $D(\mathcal{S})$ it is easy to see that the operator $A\&B : D(\mathcal{S}) \to X$ is a closed operator. Hence $D(\mathcal{S})$ is a Hilbert space in the $A\&B$-graph norm. Also one can verify that $C\&D : D(\mathcal{S}) \to Y$ is a closed operator (where $D(\mathcal{S})$ is considered with respect to the $A\&B$-graph norm). From this it follows that the system node $\mathcal{S}$ is a closed operator.

Next we show that $D(\mathcal{S})$ is dense in $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$. First observe that $\left[\begin{smallmatrix} (\alpha - A_e)^{-1}B \\ I \end{smallmatrix}\right] u \in D(\mathcal{S})$ since by the characterization of $D(\mathcal{S})$ given in Proposition 2.29 gives

$$A_e(\alpha - A_e)^{-1}Bu + Bu \quad = \quad \alpha(\alpha - A_e)^{-1}Bu \in X.$$

It is then easy to see that $\left[\begin{smallmatrix} X_1 \\ 0 \end{smallmatrix}\right] \in D(\mathcal{S})$ and hence $\left[\begin{smallmatrix} X_1 \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} (\alpha - A_e)^{-1}B \\ I \end{smallmatrix}\right] U \in D(\mathcal{S})$. From this follows that $D(\mathcal{S})$ is dense in $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ since X_1 is dense in X.

It is known (see, for instance, [MSW03]) that if the operator A is the generator of a C_0-semigroup then $\mathcal{S}$ defines a linear dynamical system as follows

Lemma 2.31 ([MSW03]): *Let $\mathcal{S}$ be a system node on (U, X, Y) as described in Definition 2.30 . Let $u \in \mathcal{C}^2([0,\infty); U)$ and $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in D(\mathcal{S})$. Then the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \mathcal{S} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0,$$

has a unique (classical) solution $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ satisfying $x \in \mathcal{C}^1([0,\infty); X) \cap \mathcal{C}^2([0,\infty); X_{-1})$, $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{C}([0,\infty); D(\mathcal{S}))$, and $y \in \mathcal{C}([0,\infty); Y)$. ♡

2.6.2. Relation of system nodes and BCS

Now that the system node and BCS were introduced, it is possible to present some results which relate both representations. The rest of this subsection can be considered as an extension of the results presented in [VZL05a].

Observe that it is not obvious how to find the B operator in the case of boundary control system as defined in Section 1.5. For those cases the lemma below gives an answer. A similar result is also given in [ET00].

Lemma 2.32: *In the case of boundary control systems as defined in Section* 1.5 *we have that the operator* B *in Definition* 2.30 *of the system node can be described by*

$$B\,u = \mathfrak{A}\mathfrak{R}u - A_e\mathfrak{R}u, \tag{2.70}$$

where $\mathfrak{A}$, $\mathfrak{R}$ *are described in Section* 1.5 *and* A_e *in Proposition* 2.29. ♡

PROOF: Let $x \in D(A^*)$ and $w(t)$, $t \geq 0$, be a solution of (1.26) with $w_0 = w(0)$ and $w_0 - \mathfrak{R}u(0) \in D(A)$. Then using equation (3.46) of [CZ95b] we have for $u \in C^2(0,t;U)$

$$\langle x, w(t)\rangle = \langle x, \mathfrak{R}\,u(t)\rangle - \langle x, \mathbb{T}(t)\,\mathfrak{R}u(0)\rangle + \langle x, \mathbb{T}(t)w_0\rangle \\ - \left\langle x, \int_0^t \mathbb{T}(t-s)\mathfrak{R}\,\dot{u}(s)\,ds\right\rangle + \left\langle x, \int_0^t \mathbb{T}(t-s)\mathfrak{A}\,\mathfrak{R}\,u(s)\,ds\right\rangle.$$

Without loss of generality we can assume that $w_0 = 0$. The above equation can also be written as follows

$$\begin{aligned}\langle x, w(t)\,\rangle =\ & \langle x, \mathfrak{R}\,u(t)\,\rangle - \langle x, \mathbb{T}(t)\,\mathfrak{R}u(0)\,\rangle - \int_0^t \langle x, \mathbb{T}(t-s)\mathfrak{R}\,\dot{u}(s)\,\rangle\ ds \\ & + \int_0^t \langle x, \mathbb{T}(t-s)\mathfrak{A}\,\mathfrak{R}\,u(s)\,\rangle\ ds \\ =\ & \langle x, \mathfrak{R}\,u(t)\,\rangle - \langle \mathbb{T}^*(t)\,x, \mathfrak{R}u(0)\,\rangle - \int_0^t \langle \mathbb{T}^*(t-s)x, \mathfrak{R}\,\dot{u}(s)\,\rangle\ ds \\ & + \int_0^t \langle \mathbb{T}^*(t-s)x, \mathfrak{A}\,\mathfrak{R}\,u(s)\,\rangle\ ds.\end{aligned}$$

By using integration by parts and the properties of the semigroup (cf. [CZ95b, §2.1]) yields

$$\begin{aligned}\langle x, w(t)\,\rangle =\ & \langle x, \mathfrak{R}\,u(t)\,\rangle - \langle \mathbb{T}^*(t)\,x, \mathfrak{R}u(0)\,\rangle - \langle \mathbb{T}^*(t-s)x, \mathfrak{R}\,u(s)\,\rangle\ \Big|_{s=0}^{s=t} \\ & - \int_0^t \langle \mathbb{T}^*(t-s)A^*\,x, \mathfrak{R}\,u(s)\,\rangle\ ds + \int_0^t \langle \mathbb{T}^*(t-s)x, \mathfrak{A}\,\mathfrak{R}\,u(s)\,\rangle\ ds \\ =\ & \langle x, \mathfrak{R}\,u(t)\,\rangle - \langle \mathbb{T}^*(t)\,x, \mathfrak{R}u(0)\,\rangle - \langle x, \mathfrak{R}\,u(t)\,\rangle + \langle \mathbb{T}^*(t)\,x, \mathfrak{R}u(0)\,\rangle \\ & - \int_0^t \langle \mathbb{T}^*(t-s)A^*\,x, \mathfrak{R}\,u(s)\,\rangle\ ds + \int_0^t \langle \mathbb{T}^*(t-s)x, \mathfrak{A}\,\mathfrak{R}\,u(s)\,\rangle\ ds \\ =\ & - \int_0^t \langle \mathbb{T}^*(t-s)A^*\,x, \mathfrak{R}\,u(s)\,\rangle\ ds + \int_0^t \langle \mathbb{T}^*(t-s)x, \mathfrak{A}\,\mathfrak{R}\,u(s)\,\rangle\ ds \\ =\ & \int_0^t \langle\langle \mathbb{T}^*(t-s)\,x, B\,u(s)\rangle\rangle_{D(A^*)}\ ds \end{aligned} \tag{2.71}$$

where $\langle\langle \cdot, \cdot \rangle\rangle_{D(A^*)}$ is the duality product of $D(A^*)$ and $D(A^*)'$, and we have defined[1]

$$\begin{aligned} \langle\langle x, B\,u \rangle\rangle_{D(A^*)} &= -\langle A^* x, \mathfrak{R}\,u\rangle + \langle x, \mathfrak{A}\mathfrak{R}\,u\rangle \\ &= -\langle x, A_e \mathfrak{R}\,u\rangle + \langle x, \mathfrak{A}\mathfrak{R}\,u\rangle . \end{aligned} \tag{2.72}$$

Now using the extension of the C$_0$-semigroup to X_{-1} we obtain from (2.71) that

$$\langle x, w(t)\,\rangle = \left\langle\left\langle x, \int_0^t \mathbb{T}_e(t-s)\,B\,u(s)\,ds \right\rangle\right\rangle_{D(A^*)}, \quad \forall\, x \in D(A^*). \tag{2.73}$$

Thus, in general, the expression above shows that $w(t)$ can be represented on X_{-1} by

$$w(t) = \mathbb{T}_e(t)w_0 + \int_0^t \mathbb{T}_e(t-s)\,B\,u(s)\,ds. \tag{2.74}$$

Since $w(t)$ is the solution of (1.26), the equation above is just another way of expressing the solutions of the boundary control problem (1.26). In fact, this formula makes sense even for $w_0 \in X_{-1}$ and $u(\cdot) \in L_2(0, t; U)$. Consequently, under these weaker assumptions, equation (2.74) can be seen as a mild solution of the following differential equation on X_{-1}

$$\dot{w}(t) = A_e w(t) + Bu(t), \qquad w(0) = w_0,\ t \geq 0.$$

The result follows from this. ∎

Remark 2.33. In [Opm05], the operator B is defined as $B = \mathfrak{A}x - A_e x$ for given $u \in U$ and a $x \in X$ such that $\mathfrak{B}x = u$. This corresponds to the same description given in (2.70) since $\mathfrak{R}$ is the right inverse of $\mathfrak{B}$. ♣

Remark 2.34. Observe that using equation (2.72) we can compute B with A^* instead of A_e, which avoids the need of finding A_e. ♣

In order to prove that this B is really the control term of a system node is necessary to prove that $B \in \mathcal{L}(U, X_{-1})$ (see Definition 2.30.(ii)). This follows easily from equation (2.72) or it can also be proved as follows.

Lemma 2.35: *The operator B described in Lemma* 2.32*, i.e., $B = -A_e\mathfrak{R} + \mathfrak{A}\mathfrak{B}$, belongs to $\mathcal{L}(U, X_{-1})$.* ♡

PROOF: This follows from the facts that $\mathfrak{R}$, $\mathfrak{A}\mathfrak{R} \in \mathcal{L}(U, X)$ and $A_e \in \mathcal{L}(X, X_{-1})$ since

$$\begin{aligned} \|Bu\|_{X_{-1}} &= \|-A_e\mathfrak{R}u + \mathfrak{A}\mathfrak{R}u\|_{X_{-1}} \\ &\leq \|-A_e\mathfrak{R}u\|_{X_{-1}} + \left\|(\alpha - A)^{-1}\mathfrak{A}\mathfrak{R}u\right\|_X \\ &\leq k_1 \|\mathfrak{R}u\|_X + k_2 \|\mathfrak{A}\mathfrak{R}u\|_X \\ &\leq k\,\|u\|_U . \end{aligned}$$

∎

[1] Observe that $\langle A^* x, w\rangle = \langle x, Aw\rangle$ only if $w \in D(A)$, if not $\langle A^* x, w\rangle = \langle\langle x, A_e w\rangle\rangle_{D(A^*)}$.

Regarding the characterization of $D(\mathcal{S})$ as described in Definition 2.30.(iii) the following result gives a convenient representation for this domain, which also avoids A_e.

Lemma 2.36: *Consider a boundary control system as defined in Section* 1.5 *and the domain* $D(\mathcal{S})$ *as described in Definition* 2.30.(iii). *Then, we have that the condition* $A_e x + Bu \in X$ *is equivalent to the condition* $(x - \mathfrak{R}u) \in X_1$. *That is*

$$A_e x + Bu \in X \quad \iff \quad x - \mathfrak{R}u \in X_1. \tag{2.75}$$

Furthermore the $A\&B$ *norm (with values in* X*) given by* (2.68) *is equivalent to the norm*

$$\|x\|_X^2 + \|u\|_U^2 + \|x - \mathfrak{R}u\|_{X_1}^2 . \qquad \heartsuit$$

PROOF: First we prove condition (2.75). Let $A_e x + Bu \in X$. Then we have that

$$\begin{aligned} & A_e x + Bu \in X \iff (\alpha - A_e)^{-1}(A_e x + Bu) \in D(A) \\ \iff \quad & (\alpha - A_e)^{-1} A_e x + (\alpha - A_e)^{-1} Bu \in D(A) \\ \iff \quad & (\alpha - A_e)^{-1} A_e x + (\alpha - A_e)^{-1}(-A_e \mathfrak{R}u + \mathfrak{A}\mathfrak{R}\, u) \in D(A) \\ \iff \quad & (\alpha - A_e)^{-1} A_e (x - \mathfrak{R}u) + (\alpha - A_e)^{-1} \mathfrak{A}\mathfrak{R}\, u \in D(A). \end{aligned} \tag{2.76}$$

Recall that $\mathfrak{A}\mathfrak{R} \in \mathcal{L}(U, X)$ and observe that when $(\alpha - A_e)^{-1}$ is restricted to X is equal to the inverse of the resolvent of the semigroup generator, $(\alpha - A)^{-1}$. We thus have that $(\alpha - A_e)^{-1}\mathfrak{A}\mathfrak{R}\, u \in D(A)$ and hence the condition (2.76) is equivalent to

$$\begin{aligned} A_e x + Bu \in X \quad & \iff \quad (\alpha - A_e)^{-1} A_e (x - \mathfrak{R}u) \in D(A) \\ & \iff \quad A_e (x - \mathfrak{R}u) \in X \\ & \iff \quad (x - \mathfrak{R}u) \in D(A) = X_1. \end{aligned}$$

Next we prove the equivalence of the norms. We have that

$$\|A_e x + Bu\|_X = \left\|(\alpha - A_e)(\alpha - A_e)^{-1}(A_e x + Bu)\right\|_X .$$

Since $(A_e x + Bu) \in X$ we have that $(\alpha - A_e)^{-1}(A_e x + Bu) \in D(A) = X_1$ and since $(\alpha - A_e)D(A) = (\alpha - A)D(A)$ we get from the equation above that

$$\begin{aligned} \|A_e x + Bu\|_X &= \left\|(\alpha - A_e)^{-1}(A_e x + Bu)\right\|_{X_1} \\ &= \left\|(\alpha - A_e)^{-1} A_e x + (\alpha - A_e)^{-1}(-A_e \mathfrak{R}u + \mathfrak{A}\mathfrak{R}u)\right\|_{X_1} \quad \text{(see (2.70))} \\ &\le \left\|(\alpha - A_e)^{-1} A_e (x - \mathfrak{R}u)\right\|_{X_1} + \|\mathfrak{A}\mathfrak{R}u\|_X \\ &= \left\|(\alpha - A_e)^{-1}(A_e - \alpha + \alpha)(x - \mathfrak{R}u)\right\|_{X_1} + \|\mathfrak{A}\mathfrak{R}u\|_X \\ &\le \|x - \mathfrak{R}u\|_{X_1} + |\alpha| \, \|x - \mathfrak{R}u\|_X + c_1 \|u\|_U \\ &\le \|x - \mathfrak{R}u\|_{X_1} + |\alpha| \, \|x\|_X + c_2 \|u\|_U . \end{aligned} \tag{2.77}$$

Now let $(x - \mathfrak{R}u) \in D(A)$ and observe that

$$\begin{aligned}
\|x - \mathfrak{R}u\|_{X_1} &= \|(\alpha - A)(x - \mathfrak{R}u)\|_X = \|(\alpha - A_e)(x - \mathfrak{R}u)\|_X \\
&= \|\alpha x - \alpha\mathfrak{R}u - A_e x + A_e\mathfrak{R}u\|_X \\
&\leq |\alpha|\,\|x\|_X + c_3\,\|u\|_U + \|A_e x - A_e\mathfrak{R}u\|_X \\
&= |\alpha|\,\|x\|_X + c_3\,\|u\|_U + \|A_e x - A_e\mathfrak{R}u + \mathfrak{A}\mathfrak{R}u - \mathfrak{A}\mathfrak{R}u\|_X \\
&\leq |\alpha|\,\|x\|_X + c_3\,\|u\|_U + \|A_e x + Bu\|_X + \|\mathfrak{A}\mathfrak{R}u\|_X \quad \text{(see (2.70))} \\
&\leq |\alpha|\,\|x\|_X + c_4\,\|u\|_U + \|A_e x + Bu\|_X\,. \qquad (2.78)
\end{aligned}$$

From equations (2.77) and (2.78) we can clearly obtain the equivalence of the two norms. ■

We have now an idea of how to construct the system node for general boundary control systems. Next we focus on the class of systems described by (2.1) and show that there is a one to one correspondence between a system node and this BCS. In particular, the following theorem shows how to represent the boundary control systems described in Theorem 2.14 as a system node.

Theorem 2.37: *Consider a BCS as described in Theorem* 2.14 *and let* $D(\mathcal{J}\mathcal{L})$ *be given by* (2.36). *Then the operator* $\mathcal{S}_b = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *described by*

$$\begin{aligned}
A\&B \begin{bmatrix} x \\ u \end{bmatrix} &= \mathcal{J}\mathcal{L}\,x \\
C\&D \begin{bmatrix} x \\ u \end{bmatrix} &= \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}
\end{aligned} \qquad (2.79)$$

with domain

$$\begin{aligned}
D(\mathcal{S}_b) &= \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \,\middle|\, \mathcal{L}x \in H^N(a,b;\mathbb{R}^n),\; W \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} = u \right\} \\
&= \begin{bmatrix} 1 \\ \mathcal{B} \end{bmatrix} D(\mathcal{J}\mathcal{L}) \qquad (2.80)
\end{aligned}$$

(where $\mathcal{B}$ *is defined in Theorem* 2.14*) is a system node. Furthermore,* $(u,\,x,\,y)$ *with* $u \in C^2(0,\infty;\mathbb{R}^{nN})$*,* $\mathcal{L}\,x(0) \in H^N(a,b;\mathbb{R}^n)$*, and* $\mathcal{B}x(0) = u(0)$ *(or equivalently,* $\left[\begin{smallmatrix} x(0) \\ u(0) \end{smallmatrix}\right] \in D(\mathcal{S}_b)$*) is a solution of the BCS in Theorem 2.14 if and only if this* $(u,\,x,\,y)$ *is a solution of the system*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \mathcal{S}_b \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0.$$

PROOF: First note that with respect to the notation of Section 1.5 we have $\mathfrak{A} = \mathcal{J}\mathcal{L}$, $D(\mathfrak{A}) = D(\mathfrak{B}) = D(\mathcal{J}\mathcal{L})$ (with $D(\mathcal{J}\mathcal{L})$ given in (2.36)), $\mathfrak{B}x = W \left[\begin{smallmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{smallmatrix}\right] = W R_{\text{ext}}\, \tau(\mathcal{L}x)$, $\mathfrak{R}$ is a right inverse of the boundary operator $\mathfrak{B}$, $U = \mathbb{R}^{nN}$, and X is given by (2.33).

We construct the system node by checking the conditions on Definition 2.30. We know that the main operator A (the semigroup generator) is obtained by setting $u = 0$, see Definition 2.30.(i). Following Theorem 2.14 we know that $A = \mathcal{A}_{\mathcal{L}} = \mathcal{J}\mathcal{L}_{|D(\mathcal{A}_{\mathcal{L}})}$, and thus condition *i* in Definition 2.30 is proved.

From Lemma 2.32 we know that for any BCS the corresponding B operator on Definition 2.30.(ii) is given by equation (2.70), see also Lemma 2.35.

Next we check condition *iii*. The domain of the system node corresponding to any BCS is given by, see Definition 2.30.(iii) and Lemma 2.36,

$$D(\mathcal{S}_b) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \Big| x - \mathfrak{R}u \in X_1 = D(\mathcal{A}_{\mathcal{L}}) \right\}.$$

But the condition $x - \mathfrak{R}u \in X_1 = D(\mathcal{A}_{\mathcal{L}})$ implies $\mathcal{L}x \in H^N(a,b)^n$ (note that $\mathfrak{R}u \in D(\mathfrak{A}) = D(\mathcal{J}\mathcal{L})$ and hence if $x - \mathfrak{R}u \in X_1$ we must have $\mathcal{L}x,\ \mathcal{L}\mathfrak{R}u \in H^N(a,b)^n$) and

$$\begin{aligned} 0 = W &\begin{bmatrix} f_{\partial,\mathcal{L}(x-\mathfrak{R}u)} \\ e_{\partial,\mathcal{L}(x-\mathfrak{R}u)} \end{bmatrix} = W R_{\text{ext}}\, \tau(\mathcal{L}(x - \mathfrak{R}u)) \\ \Rightarrow \quad & W R_{\text{ext}}\, \tau(\mathcal{L}x) = W R_{\text{ext}}\, \tau(\mathcal{L}\mathfrak{R}u) \\ \Rightarrow \quad & W \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} = u, \end{aligned}$$

since $\mathfrak{R}$ is the right inverse of $W R_{\text{ext}}\, \tau(\mathcal{L}\,\cdot)$, see Definition 1.10. From this it is clear that $D(\mathcal{S}_b)$ is given by (2.80). Also it is easy to see from the surjectivity of $\mathcal{B}$ that any $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S}_b)$ can be written as $\left[\begin{smallmatrix} 1 \\ \mathfrak{B} \end{smallmatrix}\right] D(\mathcal{J}\mathcal{L})$. Now we need to find the expression for $[A_e,\ B]_{|D(\mathcal{S}_b)}$ (recall that A_e is the extension of A to X). To find the expression for $A\&B$ observe that $\mathfrak{R} \in \mathcal{L}(U, D(\mathcal{J}\mathcal{L}))$ and $\mathfrak{B} \in \mathcal{L}(D(\mathcal{J}\mathcal{L}), U)$ satisfy $\mathfrak{B}\mathfrak{R} = I_U$. It follows that the operator $P = \mathfrak{R}\mathfrak{B}$ is a projection onto $D(\mathcal{J}\mathcal{L}) = D(\mathfrak{A})$ since $P^2 = \mathfrak{R}\mathfrak{B}\,\mathfrak{R}\mathfrak{B} = \mathfrak{R}\mathfrak{B} = P$. This gives that the operator B of Lemma 2.32 satisfies

$$B\mathfrak{B} = \mathcal{J}\mathcal{L} - A_{e\,|D(\mathfrak{A})},$$

where A_e is the extension of $\mathcal{A}_{\mathcal{L}}$ to X. Then, since any $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S}_b)$ can be written as $\left[\begin{smallmatrix} 1 \\ \mathfrak{B} \end{smallmatrix}\right] D(\mathfrak{A})$ we obtain (the top equation in (2.79))

$$A\&B \begin{bmatrix} x \\ u \end{bmatrix} = [A_e,\ B]_{|D(\mathcal{S}_b)} \begin{bmatrix} x \\ u \end{bmatrix} = A_{e\,|D(\mathfrak{A})}x + B\mathfrak{B}x = \mathcal{J}\mathcal{L}x.$$

Hence we have proved condition *iii*.

The only thing left is to prove the boundedness of the operator $C\&D$. To do so, we first prove that $\|\mathcal{L}x\|_{H^N(a,b)^n} \le \|x\|_{X_1}$ for all $x \in X_1$ (recall that $D(\mathcal{A}_\mathcal{L}) = X_1$). This follows by using the fact that $(I - \mathcal{A}) \in \mathcal{L}(H^N(a,b)^n, X)$ is a bounded bijection, and hence it has a bounded inverse in $\mathcal{L}(X, H^N(a,b)^n)$, where $\mathcal{A}$ is the operator described in Section 2.3.1. Indeed, using this we get

$$\begin{aligned}\|\mathcal{L}x\|_{H^N(a,b)^n} &= \left\|(I-\mathcal{A})^{-1}(I-\mathcal{A})\mathcal{L}x\right\|_{H^N(a,b)^n} \le c_1 \|(I-\mathcal{A})\mathcal{L}x\|_X \\ &\le c_2\big(\|x\|_X + \|\mathcal{A}\mathcal{L}x\|_X\big) \qquad \forall\, x \in X_1,\end{aligned}$$

where c_1 and c_2 are positive real constants. Note that $x \in X_1$ if and only if $\mathcal{L}x \in D(\mathcal{A})$ and that $\mathcal{A}\mathcal{L}x = \mathcal{A}_\mathcal{L}x$ for any $x \in D(\mathcal{A}_\mathcal{L})$. Thus the equation above is the same as

$$\|\mathcal{L}x\|_{H^N(a,b)^n} \le c_2\big(\|x\|_X + \|\mathcal{A}_\mathcal{L}x\|_X\big) \qquad \forall\, x \in X_1.$$

Since the graph norm is equivalent to the X_1-norm we conclude from the equation above that $\|\mathcal{L}x\|_{H^N(a,b)^n} \le c\,\|x\|_{X_1}$ for all $x \in X_1$ and some $c \in \mathbb{R}_+$. Using this we obtain for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S})$

$$\begin{aligned}\left\|C\&D\begin{bmatrix} x \\ u\end{bmatrix}\right\|_U &= \left\|\widetilde{W}R_{\text{ext}}\,\tau(\mathcal{L}x)\right\|_U \le \|\mathcal{L}x\|_{H^N(a,b)^n} \\ &\le c_1\big(\|\mathcal{L}(x-\mathfrak{R}u)\|_{H^N(a,b)^n} + \|\mathcal{L}\mathfrak{R}u\|_{H^N(a,b)^n}\big) \\ &\le c\big(\|x-\mathfrak{R}u\|_{X_1} + \|u\|_U\big),\end{aligned}$$

where we used the boundedness of the boundary trace operator, see Theorem 2.6, and the fact that $(x - \mathfrak{R}u) \in X_1$. The desired result follows now from Lemma 2.36.

The final claim about the equivalence of solutions follows immediately from equation (2.79), see [MS06]. ■

Chapter 3

Energy Preserving and Conservative Systems

In the previous chapter we described how to obtain energy preserving systems for a class of boundary control systems related to a skew-symmetric operator $\mathcal{J}$, see Theorem 2.14. In this chapter we continue studying the same class of systems. However, we mainly focus on impedance and scattering energy preserving systems as described in Theorems 2.16 and 2.17. In particular we show that they are in fact conservative systems using the concept of a system node. Also, we investigate the relation between stability, controllability and observability for these two classes, which will help us to prove this properties in the forthcoming chapters. We start by briefly introducing the concepts of observability, controllability and a stronger version of well-posedness. Then, we study impedance energy preserving systems as obtained in the previous chapter, followed by scattering energy preserving systems. In particular, we show that those two classes of systems (as obtained in Chapter 2) are also conservative. Finally, we describe how we could deal with other other supply rates.

In this chapter we follow closely the notation of Section 2.6.1. In particular, we write A for the main operator of the system node, as well as its extension A_e, see Proposition 2.29. The same applies to the semigroup $\mathbb{T}$.

3.1. Observability, controllability and well-posedness

In this subsection we briefly describe some concepts that are important in the theory of linear infinite dimensional systems. These notions presented here are all well known and can be found easily in the literature. For more details and information we refer, for instance, to [WT03], [TW03], [JZ02a], or [Sta05]. We start with the notion of well-posed systems. The class of well-posed infinite dimensional linear systems was first introduced by Salamon and Weiss in the late 80's, see [Sal89] or [Wei89]. The aim was to provide a mathematical framework for the systematic description, analysis and control of systems described by partial differential equations. The definition below is an adaptation to the class of boundary control systems studied here. Roughly speaking a system is well-posed (in the sense of Weiss and Salamon) if for every locally square integrable input and every initial condition there exists a well-defined solution such that the output is locally square integrable. For our class of systems this is formalized in the following definition. (Note that $D(\mathcal{A}_{\mathcal{L}}) \subset D(\mathcal{J}\mathcal{L})$, where $D(\mathcal{J}\mathcal{L})$ is given by (2.36).)

Definition 3.1. Consider the system described in Theorem 2.14 and let $t_f > 0$. We say that the system is *well-posed in the sense of Weiss and Salamon* if the following holds:

- The operator $\mathcal{A}_{\mathcal{L}}$ is the infinitesimal generator of a C_0-semigroup on the space $L^2(a, b; \mathbb{R}^n)$.
- There exists an $m > 0$ such that the following inequality holds for all $u \in C^2([0, t_f); \mathbb{R}^n)$ and $x_0 \in D(\mathcal{J}\mathcal{L})$ such that $u(0) = \mathcal{B}x_0$

$$\|x(t_f)\|_{\mathcal{L}}^2 + \int_0^{t_f} \|y(t)\|^2 dt \leq m \left[\|x_0\|_{\mathcal{L}}^2 + \int_0^{t_f} \|u(t)\|^2 dt \right]. \tag{3.1}$$

♣

For brevity we say that a system is well-posed when no confusion may arise. Since the spaces $D(\mathcal{J}\mathcal{L})$ and $C^2([0, t_f); \mathbb{R}^n)$ are dense linear subspaces of the spaces $L^2(a, b; \mathbb{R}^n)$ and $L^2([0, t_f); \mathbb{R}^n)$, respectively, we find that if (3.1) holds for all $x_0 \in D(\mathcal{J}\mathcal{L})$ and $r \in C^2([0, t_f), \mathbb{R}^n)$, then (3.1) holds for all $x_0 \in L^2(a, b; \mathbb{R}^n)$ and $r \in L^2([0, t_f); \mathbb{R}^n)$. Hence if the system is well-posed, then it has for every initial condition and every square integrable input a unique (mild) solution, and (3.1) still holds.

Recall that the *growth bound* of a strongly continuous semigroup $\mathbb{T}$ is

$$\omega_0(\mathbb{T}) = \lim_{t \to \infty} \frac{1}{t} \log \|\mathbb{T}(t)\| = \inf_{t > 0} \frac{1}{t} \log \|\mathbb{T}(t)\| .$$

In this case, for all $\omega > \omega_0(\mathbb{T})$ there exists a constant M_ω such that $\forall\, t \geq 0$ the semigroup $\mathbb{T}$ satisfies $\|\mathbb{T}(t)\| \leq M_\omega \mathrm{e}^{\omega t}$, see for example [CZ95b, §2.1].

Definition 3.2. The semigroup $\mathbb{T}$ on a Hilbert space X is *exponentially stable* if its growth bound is negative, i.e., $\omega_0(\mathbb{T}) < 0$. Equivalently, there exist positive constants M and α such that $\|\mathbb{T}(t)\| \leq M\mathrm{e}^{-\alpha t}$. $\mathbb{T}$ is *asymptotically (or strongly) stable* if

$$\lim_{t \to \infty} \|\mathbb{T}(t)x_0\| = 0 \qquad \forall\, x_0 \in X.$$

Finally, $\mathbb{T}$ is *weakly stable* if $\lim_{t \to \infty} \langle \mathbb{T}(t)x_0, y_0 \rangle = 0$, for all $x_0,\, y_0 \in X$. ♣

Next we introduce the concept of admissible control and observation operators.

Definition 3.3 (Jacob and Zwart [JZ02b]). Following the notation of Section 2.6.1 we define the following operators.

(1) Let $B \in \mathcal{L}(U, X_{-1})$. For $t \geq 0$ we define the operator $\mathbb{B}_t : L_2(0,\infty;U) \to X_{-1}$ by

$$\mathbb{B}_t u := \int_0^t \mathbb{T}(t-\tau)Bu(\tau)\,d\tau.$$

Then B is called an *admissible control operator* for $\mathbb{T}(t)$ if, for some (and hence for any) $t > 0$, $\mathbb{B}_t \in \mathcal{L}(L_2(0,\infty;U), X)$.

(2) Let B be an admissible control operator for $\mathbb{T}(t)$. B is called an *infinite-time admissible control operator* for $\mathbb{T}(t)$ if $\mathbb{T}(\cdot)Bu(\cdot) : [0,\infty) \to X_{-1}$ is integrable for every $u \in L_2(0,\infty;U)$, and the operator $\mathbb{B}_\infty : L_2(0,\infty;U) \to X_{-1}$, given by

$$\mathbb{B}_\infty u := \lim_{t\to\infty} \int_0^t \mathbb{T}(\tau)Bu(\tau)\,d\tau,$$

satisfies $\mathbb{B}_\infty \in \mathcal{L}(L_2(0,\infty;U), X)$.

(3) Let $C \in \mathcal{L}(X_1, Y)$. Then C is called an *admissible observation operator* for $\mathbb{T}(t)$ if, for some (and hence any) $t > 0$, there is some $K > 0$ such that

$$\|C\mathbb{T}(\cdot)x_0\|_{L_2(0,t)} \leq K\,\|x_0\|_X\,, \qquad \forall\, x_0 \in D(A).$$

(4) Let C be an admissible observation operator for $\mathbb{T}(t)$. We call C an *infinite-time admissible observation operator* if there is some $K > 0$ such that

$$\|C\mathbb{T}(\cdot)x_0\|_{L_2(0,\infty)} \leq K\,\|x_0\|_X\,, \qquad \forall\, x_0 \in D(A). \qquad ♣$$

In other words, B is an admissible control operator if the following property holds: If $x \in X_{-1}$ is a solution of $\dot{x}(t) = A_e x(t) + Bu(t)$ with $x(0) = x_0 \in X$ and $u \in L_2(0,\infty;U)$, then $x(t) \in X$ for all $t \geq 0$. In this case, x is a continuous X-valued function of t and

$$x(t) = \mathbb{T}(t)x_0 + \mathbb{B}_t u,$$

where $\mathbb{B}_t$ is defined by (compare with equation (2.73))

$$\mathbb{B}_t u := \int_0^t \mathbb{T}(t-\tau)Bu(\tau)\,d\tau.$$

The above integration is done in X_{-1}, but the result is in X. The admissibility of B is also equivalent to the range of $\mathbb{B}_t$ being in X for some $t > 0$. If $\mathbb{T}(t)$ is exponentially stable, then the two notions of admissibility coincide. Moreover, B is an (infinite-time) admissible control operator for $\mathbb{T}(t)$ if and only if B^* is an (infinite-time) admissible observation operator for $\mathbb{T}^*(t)$.

Definition 3.4. In the following we use the notation of Section 2.6.1. Let B be an admissible control operator for $\mathbb{T}(t)$, then:

(1) $(\mathbb{T}, B)$ is *exactly controllable in finite time* if there exists a time t_0 such that $\operatorname{ran} \mathbb{B}_{t_0} = X$.

(2) $(\mathbb{T}, B)$ is *approximately controllable* if $\bigcup_{t \geq 0} \operatorname{ran} \mathbb{B}_t$ is dense in X.

(3) $(\mathbb{T}, B)$ is *exactly controllable in infinite time* if B is infinite-time admissible for $\mathbb{T}(t)$ and $\operatorname{ran} \mathbb{B}_\infty = X$. ♣

Observe that the definition above corresponds to the usual notion of being able to steer any initial state to a desired final state (or close to it). Next we introduce the corresponding observability concepts via duality.

Definition 3.5. Suppose that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}(t)$ (Equivalently, C^* is an admissible control operator for the adjoint semigroup $\mathbb{T}^*$). We say that $(\mathbb{T}, C)$ is *exactly observable* (in time t_0) (in infinite time) if $(\mathbb{T}^*, C^*)$ is exactly controllable (in time t_0) (in infinite time). Similarly, $(\mathbb{T}, C)$ is *approximately observable* (in time t_0) (in infinite time) if $(\mathbb{T}^*, C^*)$ is approximately controllable (in time t_0) (in infinite time). ♣

If any of the operators $(\mathbb{T}, B, C)$ satisfies any of the properties described above, then we say that the system has that property. That is why we introduce the following definition.

Definition 3.6. A linear system is called exponentially (asymptotically, weakly) stable if $\mathbb{T}$ is an exponentially (asymptotically, weakly) stable C_0-semigroup. The system is called infinite time admissible if B is an infinite-time admissible control operator and C is an infinite-time admissible observation operator for $\mathbb{T}(t)$. Furthermore, it will be called approximately controllable, exactly controllable, or exactly controllable in infinite time if $(\mathbb{T}, B)$ has that property. Similarly for the observability property. ♣

The following definition of observability is equivalent to the one introduced in Definition 3.5, see [RTTT05] or [TW03] and the references therein for details on this equivalence.

Definition 3.7. Consider a system as described in Theorem 2.14. Then the system is *exactly observable* in time t_f if there exist $k_{t_f} > 0$ such that

$$\int_0^{t_f} \|y(t)\|_Y^2 \, dt \geq k_{t_f}^2 \|x_0\|_{\mathcal{L}}^2, \qquad \forall \, x_0 \in D(\mathcal{A}_{\mathcal{L}}). \tag{3.2}$$

Note that $y(t) = C\mathbb{T}(t)x_0$. The system is exactly observable if it is exactly observable in some time $t_f > 0$. The system is *exactly observable in infinite time* if and only if C is infinite time admissible and the equation (3.2) above holds for $t_f = \infty$. The system is *approximately observable* in time t_f (or infinite time) if and only if $C\mathbb{T}(t)x_0 = 0$ for $t \in [0, \, t_f]$ implies $x_0 = 0$. ♣

3.2. Impedance passive system nodes and BCS

In Section 2.6.2 we have seen how to describe BCS by a system node. We also showed how to obtain energy preserving systems out of the class of systems (2.1). In this section we focus on impedance energy preserving systems and we show some properties of the system node in this case. As mentioned earlier, writing a boundary control system as a system node allows us to use results available for systems nodes to analyze BCS.

Definition 3.8 (O.J. Staffans [Sta02]). A system node $\mathcal{S}$ on (U, X, U) (note that $U = Y$) is *impedance passive* if, for all $t > 0$, the solution (x, y) in Lemma 2.31 satisfies

$$\|x(t)\|_X^2 - \|x_0\|_X^2 \leq 2\int_0^t \text{Re}\,\langle y(\tau), u(\tau)\rangle_U \, d\tau. \tag{3.3}$$

It is *impedance energy preserving* if the above inequality hold in the form of an equality. Finally, $\mathcal{S}$ is *impedance conservative* if both $\mathcal{S}$ and the dual system node $\mathcal{S}^*$ are impedance energy preserving. ♣

For a definition of the dual (also called the unbounded adjoint) of the system node see, for instance, [Sta05], [MSW03] or [Sta02]. The following two theorems characterize impedance energy preserving and impedance conservative system nodes, respectively. For the proof we refer to [Sta02].

Theorem 3.9 (O.J. Staffans [Sta02]): *Let* $\mathcal{S} = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a system node on* (U, X, U)*. Then the following conditions are equivalent:*

(i) $\mathcal{S}$ *is impedance energy preserving.*

(ii) For all $t > 0$*, the solution* (x, y) *in Lemma* 2.31 *satisfies*

$$\frac{d}{dt}\|x(t)\|_X^2 = 2\text{Re}\,\langle y(t), u(t)\rangle_U\,.$$

(iii) For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in D(\mathcal{S})$*,*

$$\text{Re}\left\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}\right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} = 0.$$

(vi) The system node $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is skew-symmetric, i.e.,* $D(\mathcal{S}) = D\left(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]\right) \subset D\left(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]^*\right)$*, and*

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}^* \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = -\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in D(\mathcal{S}).$$

Note that if $\mathcal{S} = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a system node, then so is $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$, and that the domains of these two nodes are the same (it depends only on $A\&B$). In the next theorem we use the notion of the *transfer function* of a system node $\mathcal{S} = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, which can be defined by (see [MSW03])

$$G(s) := C\&D \begin{bmatrix} (s - A_e)^{-1}B \\ I \end{bmatrix}, \qquad s \in \rho(A), \tag{3.4}$$

which is an $\mathcal{L}(U, Y)$-valued analytic function.

Theorem 3.10 (O.J. Staffans [Sta02]): *Let $\mathcal{S} = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, U) with transfer function G. Then the following conditions are equivalent:*

(i) $\mathcal{S}$ is impedance conservative.

(ii) For all $t > 0$, the solution (x, y) in Lemma 2.31 satisfies

$$\frac{d}{dt} \|x(t)\|_X^2 = 2Re \, \langle y(t), u(t) \rangle_U \, .$$

and the same identity is true for the adjoint system.

(iii) The system node $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is skew-adjoint, i.e., $D(\mathcal{S}) = D\left(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]\right) = D\left(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]^*\right)$, and

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}^* = -\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}.$$

(iv) $A = -A^*$, $B^* = C$, and $G(\alpha) + G(-\overline{\alpha})^* = 0$ for some (or equivalently, for all) $\alpha \in \rho(A)$ (in particular, this identity is valid for all α with $Re\, \alpha \neq 0$).

Next we study the class of boundary control systems described in Theorem 2.14. Below we show that if these systems are impedance energy preserving (as described in Theorem 2.16), then they are impedance conservative. That is, its dual boundary control system is also impedance energy preserving.

Theorem 3.11: *Given a system node $\mathcal{S} = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ as described in Theorem 2.37, which is impedance energy preserving, i.e., $\frac{1}{2}\frac{d}{dt} \|x(t)\|_{\mathcal{L}}^2 = u(t)^T y(t)$ holds (see Theorem 2.16). Then, it is also impedance conservative. As a consequence, $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is skew-adjoint, $A = -A^* = \mathcal{A}_{\mathcal{L}}$, $B^* = C = (C\&D)_{|X_1}$, and $G(\alpha) + G(-\overline{\alpha})^* = 0$ for some (or equivalently, for all) $\alpha \in \rho(A)$.*

PROOF: Recall that the state space X is described in (2.33) and here we denote its inner product by $\langle\cdot,\cdot\rangle_X$, i.e., $\langle\cdot,\cdot\rangle_X = \langle\cdot,\cdot\rangle_{\mathcal{L}}$. If we prove that the system node $\mathcal{S}_1 = \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is skew-adjoint, then the result will follow from Theorem 3.10. So, we find the adjoint of the system node as follows. Let $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S}_1) = D(\mathcal{S})$ and $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$. Then we have that

$$\left\langle \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} = \left\langle A\&B \begin{bmatrix} x \\ u \end{bmatrix}, v \right\rangle_X - \left\langle C\&D \begin{bmatrix} x \\ u \end{bmatrix}, w \right\rangle_U .$$

Using Theorem 2.37 yields

$$\left\langle \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} = \langle \mathcal{J}\mathcal{L}x, v\rangle_X - \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, w \right\rangle_U . \tag{3.5}$$

First we want to prove that $\mathcal{L}v \in H^N(a,b)^n$, i.e., $v \in D(\mathcal{J}\mathcal{L})$ with $D(\mathcal{J}\mathcal{L})$ given in (2.36), see equation (2.80). Choosing x such that $\mathcal{L}x \in H^N(a,b)^n$ has compact support strictly included in (a,b), yields that u and y are zero and hence, in this case, we get

$$\left\langle \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} = \langle \mathcal{J}\mathcal{L}x, v\rangle_X = \int_a^b (\mathcal{J}\mathcal{L}x(z))^T \mathcal{L}v(z)\, dz.$$

Since the equality above holds for every $\mathcal{L}x \in H^N(a,b)^n$ with compact support strictly included in (a,b), we must have (by the definition of the adjoint operator) that every $\mathcal{L}v$ is N-times differentiable, that is $\mathcal{L}v \in H^N(a,b)^n$.

Now we can use (2.35) in equation (3.5), which gives

$$\begin{aligned}\left\langle \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} &= -\langle x, \mathcal{J}\mathcal{L}v\rangle_X + \left\langle \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, \Sigma \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U \\ &\quad - \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, w \right\rangle_U ,\end{aligned}$$

and using equation (2.49) yields

$$\begin{aligned}\left\langle \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} &= -\langle x, \mathcal{J}\mathcal{L}v\rangle_X \\ + \left\langle \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, (W^T\widetilde{W} + \widetilde{W}^T W) \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U &- \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, w \right\rangle_U .\end{aligned}$$

Since $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S})$ we obtain from Theorem 2.37 and the equation above

$$\begin{aligned}
\left\langle \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} &= -\langle x, \mathcal{J}\mathcal{L}v\rangle_X + \left\langle u, \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U \\
&+ \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, W \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U - \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, w \right\rangle_U \\
&= \langle x, -\mathcal{J}\mathcal{L}v\rangle_X + \left\langle u, \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U + \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, W \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} - w \right\rangle_U . \\
&= \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} -\mathcal{J}\mathcal{L}v \\ \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} + \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, W \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} - w \right\rangle_U .
\end{aligned}$$

Since this holds for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S})$ we conclude by using the definition of the adjoint operator, see for instance (2.63), that

$$(A\&B)^* \begin{bmatrix} v \\ w \end{bmatrix} = -\mathcal{J}\mathcal{L}v, \quad (-C\&D)^* \begin{bmatrix} v \\ w \end{bmatrix} = \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} = C\&D \begin{bmatrix} v \\ w \end{bmatrix}$$

and that

$$\widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} \text{ is orthogonal to } \left(W \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} - w \right). \tag{3.6}$$

Note that for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S})$, see (2.80), $\widetilde{W} \left[\begin{smallmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{smallmatrix}\right]$ maps all $\mathbb{R}^{nN}$ (by the surjectivity of $\widetilde{W}$ and of the boundary operator $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$, see the proof of Theorem 2.14). Thus, it is easy to see that the condition (3.6) above implies $W \left[\begin{smallmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{smallmatrix}\right] - w = 0$. Hence the domain of $\mathcal{S}_1^*$ can be represented by

$$D(\mathcal{S}_1^*) = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \middle| \; \mathcal{L}v \in H^N(a,b)^n \text{ and } W \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} = w \right\} = D(\mathcal{S}).$$

From this (see Theorem 2.37) we clearly obtain the desired result. Note that the domain of $\mathcal{S}_1$ and $\mathcal{S}$ are the same, since the domain of $\mathcal{S}$ only depends on A and B, see Definition 2.30. ■

It is possible to verify that, in the impedance energy preserving case, there is relation between the graph of the system node (with $\mathcal{L} = I$) and the Dirac structure described in Theorem 2.7. For instance, by letting $W = [I, 0]$ (which satisfies $W\Sigma W^T = 0$) and $\widetilde{W} = [0, I]$ in Theorem 2.10 we obtain an impedance energy preserving system (with $\mathcal{L} = I$). In this case, the graph of the system operator $\mathcal{S}$, see Theorem 2.37, is given by

$$G_{\mathcal{S}} = \begin{bmatrix} A\&B \\ C\&D \\ I_X \quad 0 \\ 0 \quad I_U \end{bmatrix} D(\mathcal{S})$$
$$= \left\{ \begin{bmatrix} f \\ y \\ e \\ u \end{bmatrix} \in \mathcal{B} \;\middle|\; e \in H^N(a,b;\mathbb{R}^n),\ \mathcal{J}e = f,\ \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} R_{\text{ext}}\,\tau(e) \right\}.$$

Comparing this with (2.15) we can see that the two subspaces are equivalent. One just swaps the position of u and y to obtain the subspace (2.15). This result can be generalized for impedance energy preserving BCS described in Theorem 2.16 as the following theorem shows.

Theorem 3.12: *Consider a Dirac structure $D_{\mathcal{J}}$ as described in Theorem 2.7 and an impedance energy preserving BCS with $\mathcal{L} = I$ as described in Theorem 2.16 with corresponding system node $\mathcal{S}$ given in Theorem 2.37. Then, there exist a matrix $U = \left[\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right]$ satisfying $U^T \Sigma U = \Sigma$ such that*

$$MG_{\mathcal{S}} = D_{\mathcal{J}}, \qquad M = \begin{bmatrix} I_X & 0 & 0 & 0 \\ 0 & U_1 & 0 & U_2 \\ 0 & 0 & I_X & 0 \\ 0 & U_3 & 0 & U_4 \end{bmatrix}$$

where $G_{\mathcal{S}}$ is the graph of the system node. Hence, the Dirac structure can be seen as the graph of a closed operator (since $\mathcal{S}$ is closed). As a consequence, $\mathcal{D}_{\mathcal{J}}$ is a closed subspace of the bond space $\mathcal{B}$.

PROOF: First notice from the proof of Theorem 2.16, see (2.50), that we also have

$$\begin{bmatrix} \widetilde{W} \\ W \end{bmatrix}^{-T} \Sigma \begin{bmatrix} \widetilde{W} \\ W \end{bmatrix}^{-1} = \Sigma \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^{-T} \Sigma \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^{-1} \Sigma = \Sigma. \tag{3.7}$$

The results follows easily by letting $U = \left[\begin{smallmatrix} \widetilde{W} \\ W \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right]$ and noticing that the graph of $\mathcal{S}$ is

$$G_{\mathcal{S}} = \left\{ \begin{bmatrix} f \\ y \\ e \\ u \end{bmatrix} \in \mathcal{B} \;\middle|\; e \in H^N(a,b;\mathbb{R}^n),\ \mathcal{J}e = f,\ \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \widetilde{W} \\ W \end{bmatrix} R_{\text{ext}}\,\tau(e) \right\}. \blacksquare$$

3.3. Scattering energy preserving systems

In this section we study scattering boundary control systems. Scattering systems are characterized by the fact that at any time, the sum of the final energy and the output energy cannot be larger than the sum of the initial energy and the input energy. Thus, they are well-posed in the sense of Weiss (see Definition 3.1), as can be seen from the following definition.

Definition 3.13 (O.J. Staffans, [Sta05]). A system node $\mathcal{S}$ on the spaces (U, X, Y) is K*-scattering passive* if, for all $t > 0$, the solution (x, y) in Lemma 2.31 satisfies

$$\|x(t)\|_X^2 - \|x_0\|_X^2 \leq K \int_0^t \|u(\tau)\|_U^2 \, d\tau - K \int_0^t \|y(\tau)\|_Y^2 \, d\tau. \tag{3.8}$$

It is K*-scattering energy preserving* if the above inequality hold in the form of an equality. Finally, $\mathcal{S}$ is *scattering conservative* if $\mathcal{S}$ is K-scattering energy preserving and the dual system node $\mathcal{S}^*$ is K^{-1}-scattering energy preserving. ♣

For simplicity we only consider 2-scattering systems in this book and we simple say scattering systems. Recall that in our setting we have $U = Y = \mathbb{R}^{nN}$, and X is defined in (2.33). Note also that equation (3.8) is equivalent to the one appearing in Theorem 2.17. The following theorem describes the dual system of a scattering boundary control system, and it also shows that this class of systems is conservative.

Theorem 3.14: *Let* $\mathcal{S} = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a system node as described in Theorem 2.37. Assume that it is 2-scattering energy preserving, i.e.,* $\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \|u(t)\|_U^2 - \|y(t)\|_Y^2$ *holds (see Theorem 2.17). Then, the adjoint system node is described by*

$$A\&B^* \begin{bmatrix} x \\ u \end{bmatrix} = -\mathcal{J}\mathcal{L}x$$

$$D(\mathcal{S}^*) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \,\middle|\, \mathcal{L}x \in H^N(a, b; \mathbb{R}^n),\ 2\widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} = u \right\},$$

$$C\&D^* \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = 2W \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}.$$

Furthermore, the adjoint system is 2^{-1}*-scattering energy preserving, i.e., the system is scattering conservative. In addition, the adjoint boundary control* *system is described by*

$$\begin{aligned}\frac{\partial v}{\partial t}(t) &= -\mathcal{J}\mathcal{L}v(t), \quad v(0)=v_0,\ t\geq 0,\\ \tilde{u}(t) &= 2\widetilde{W}\begin{bmatrix} f_{\partial,\mathcal{L}v}(t)\\ e_{\partial,\mathcal{L}v}(t)\end{bmatrix},\\ \tilde{y}(t) &= 2W\begin{bmatrix} f_{\partial,\mathcal{L}v}(t)\\ e_{\partial,\mathcal{L}v}(t)\end{bmatrix},\end{aligned}$$

with semigroup generator $\mathcal{A}_{\mathcal{L}}^*$, where $\mathcal{A}_{\mathcal{L}}$ is the semigroup generator of the original system.

PROOF: Recall that the state space X is described in (2.33) and here we denote its inner product by $\langle\cdot,\cdot\rangle_X$, i.e., $\langle\cdot,\cdot\rangle_X = \langle\cdot,\cdot\rangle_{\mathcal{L}}$. First we find the adjoint of the system node as it was done in the proof of theorem 3.11. Let $\left[\begin{smallmatrix} x\\ u\end{smallmatrix}\right] \in D(\mathcal{S})$ and $\left[\begin{smallmatrix} v\\ w\end{smallmatrix}\right] \in \left[\begin{smallmatrix} X\\ U\end{smallmatrix}\right]$. Then we have that

$$\left\langle \begin{bmatrix} A\&B\\ C\&D\end{bmatrix}\begin{bmatrix} x\\ u\end{bmatrix}, \begin{bmatrix} v\\ w\end{bmatrix}\right\rangle_{\left[\begin{smallmatrix} X\\ U\end{smallmatrix}\right]} = \left\langle A\&B\begin{bmatrix} x\\ u\end{bmatrix}, v\right\rangle_X + \left\langle C\&D\begin{bmatrix} x\\ u\end{bmatrix}, w\right\rangle_U.$$

Using Theorem 2.37 yields

$$\left\langle \begin{bmatrix} A\&B\\ C\&D\end{bmatrix}\begin{bmatrix} x\\ u\end{bmatrix}, \begin{bmatrix} v\\ w\end{bmatrix}\right\rangle_{\left[\begin{smallmatrix} X\\ U\end{smallmatrix}\right]} = \langle \mathcal{J}\mathcal{L}x, v\rangle_X + \left\langle \widetilde{W}\begin{bmatrix} f_{\partial,\mathcal{L}x}\\ e_{\partial,\mathcal{L}x}\end{bmatrix}, w\right\rangle_U. \tag{3.9}$$

Again, choosing an $\mathcal{L}x \in H^N(a,b)^n$ with compact support as we did in the proof of Theorem 3.11, yields $\mathcal{L}v \in H^N(a,b)^n$. Now, we can use (2.35) in equation (3.9), which gives

$$\begin{aligned}\left\langle \begin{bmatrix} A\&B\\ C\&D\end{bmatrix}\begin{bmatrix} x\\ u\end{bmatrix}, \begin{bmatrix} v\\ w\end{bmatrix}\right\rangle_{\left[\begin{smallmatrix} X\\ U\end{smallmatrix}\right]} &= -\langle x, \mathcal{J}\mathcal{L}v\rangle_X + \left\langle \begin{bmatrix} f_{\partial,\mathcal{L}x}\\ e_{\partial,\mathcal{L}x}\end{bmatrix}, \Sigma\begin{bmatrix} f_{\partial,\mathcal{L}v}\\ e_{\partial,\mathcal{L}v}\end{bmatrix}\right\rangle_U\\ &\quad + \left\langle \widetilde{W}\begin{bmatrix} f_{\partial,\mathcal{L}x}\\ e_{\partial,\mathcal{L}x}\end{bmatrix}, w\right\rangle_U,\end{aligned}$$

and using equation (2.52) yields that this is equal to

$$\begin{aligned}\left\langle \begin{bmatrix} A\&B\\ C\&D\end{bmatrix}\begin{bmatrix} x\\ u\end{bmatrix}, \begin{bmatrix} v\\ w\end{bmatrix}\right\rangle_{\left[\begin{smallmatrix} X\\ U\end{smallmatrix}\right]} &= -\langle x, \mathcal{J}\mathcal{L}v\rangle_X\\ + \left\langle \begin{bmatrix} f_{\partial,\mathcal{L}x}\\ e_{\partial,\mathcal{L}x}\end{bmatrix}, 2(W^TW - \widetilde{W}^T\widetilde{W})\begin{bmatrix} f_{\partial,\mathcal{L}v}\\ e_{\partial,\mathcal{L}v}\end{bmatrix}\right\rangle_U &+ \left\langle \widetilde{W}\begin{bmatrix} f_{\partial,\mathcal{L}x}\\ e_{\partial,\mathcal{L}x}\end{bmatrix}, w\right\rangle_U.\end{aligned}$$

Since $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S})$ we obtain from Theorem 2.37 and the equation above

$$\begin{aligned}
\left\langle \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} &= -\langle x, \mathcal{J}\mathcal{L}v\rangle_X + \left\langle u, 2W \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U \\
&\quad + \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, -2\widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U + \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, w \right\rangle_U \\
&= \langle x, -\mathcal{J}\mathcal{L}v\rangle_X + \left\langle u, 2W \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U \\
&\quad + \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, -2\widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} + w \right\rangle_U .
\end{aligned}$$

Again, as it was done at the end of the proof of theorem 3.11 we can conclude from this, together with the surjectivity of the boundary operator, and the definition of the adjoint operator that

$$(A\&B)^* \begin{bmatrix} v \\ w \end{bmatrix} = -\mathcal{J}\mathcal{L}v, \quad (C\&D)^* \begin{bmatrix} v \\ w \end{bmatrix} = 2W \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix}$$

and that

$$D(\mathcal{S}^*) = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \,\middle|\, \mathcal{L}v \in H^N(a,b)^n \text{ and } 2\widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} = w \right\}.$$

From this (see Theorem 2.37) we clearly obtain the desired result. By the definition of the system node, the corresponding semigroup generator of $\mathcal{S}^*$, say A^*, is given, in this case, by $A^* = -\mathcal{J}\mathcal{L}$ with domain $D(\mathcal{S}^*)_{|u=0}$, i.e.,

$$A^* = -\mathcal{J}\mathcal{L}, \quad D(A^*) = \left\{ v \in H^N(a,b)^n \mid \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \in \ker \widetilde{W} \right\}.$$

Since $\widetilde{W}$ satisfies (2.51), it follows from Theorem 2.24 that A^* (and its domain) equals the adjoint of the semigroup generator (of the original system) $\mathcal{A}_{\mathcal{L}}$, i.e., $A^* = \mathcal{A}_{\mathcal{L}}^*$, and hence, it also generates a contraction semigroup. From this we deduce the expression for the adjoint boundary control system.

The only thing left to prove is that the system is conservative, i.e., that the dual system is also (2^{-1})-scattering energy preserving. This follows from

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v(t)\|_{\mathcal{L}}^2 = \frac{1}{2}\frac{d}{dt}\langle v(t), v(t)\rangle_X &= -\langle \mathcal{J}\mathcal{L}v(t), v(t)\rangle_X \\
&= -\frac{1}{2}\left\langle \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix}, \Sigma \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix} \right\rangle_U \\
&= -\left\langle \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix}, (W^TW - \widetilde{W}^T\widetilde{W}) \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix} \right\rangle_U \\
&= -\frac{1}{4}\|\tilde{y}(t)\|_U^2 + \frac{1}{4}\|\tilde{u}(t)\|_U^2,
\end{aligned}$$

where we used (2.35) and (2.52). ■

Remark 3.15. It is worth remarking that, following Theorem 6.2.13 of [Sta05], the operator $\mathcal{S}^*$ corresponds to a representation of the *dual system* of the system represented by $\mathcal{S}$ and hence of the BCS. For more details on this see [Sta05] or [SW04]. ♣

Notice that, in this case, the dual BCS corresponds to a system whose input and output have been interchanged (with respect to the original BCS). A system with this property is called flow-invertible, see [SW04].

The following theorem links exponential stability, observability and controllability of the class of scattering energy preserving systems described in the previous theorem. This result is related to Proposition 3.2 of [TW03].

Theorem 3.16: *Consider a* 2*-scattering energy preserving BCS as described in Theorem* 2.17*. Then the following statements are equivalent:*

(a) The system is exactly controllable in time t_f*, hence for all* $t > t_f$*.*

(b) The system is exactly observable in time t_f*, hence for all* $t > t_f$*.*

(c) The semigroup $\mathbb{T}$ *generated by* $\mathcal{A}_{\mathcal{L}}$ *satisfies* $\|\mathbb{T}(t_f)\| < 1$ *(in particular, the system is exponentially stable).*

PROOF: (c) ⇒ (b): Since the system is scattering energy preserving it satisfies for all $x_0 \in D(\mathcal{A}_{\mathcal{L}})$ and $u(t) = 0$

$$\|x(t_f)\|_X^2 + 2\int_0^{t_f} \|y(\tau)\|_Y^2 \, d\tau = \|x_0\|_X^2 \, . \tag{3.10}$$

If (c) holds, then $\|\mathbb{T}(t_f)\| = (1 - k_T^2)$ for some $k_T > 0$, and hence $\|x(t_f)\|_X = \|\mathbb{T}(t_f)x_0\|_X \le (1 - k_T^2)\|x_0\|_X$. This together with equation (3.10) implies that $\int_0^{t_f} \|y(\tau)\|_Y^2 \, d\tau \ge k_T^2 \|x_0\|_X^2$ for some $k_T > 0$. The exact observability now follows from Definition 3.7.

(b) ⇒ (c): The same procedure done backwards.

(a) $\iff$ (c): (c) is equivalent to the fact that $\|\mathbb{T}^*\| < 1$. (a) is equivalent to the dual system being exactly observable (see Remark 3.15) in time t_f. Since the dual system is also scattering energy preserving according to Theorem 3.14, the result now follows from the equivalence of (b) and (c) proved earlier. ■

The following theorem is related to Proposition 3.4 of [TW03].

Theorem 3.17: *Consider a scattering energy preserving BCS as described in Theorem 2.17. Then the following statements are equivalent:*

(a) $\mathbb{T}$ *is asymptotically stable.*

(b) The system is exactly observable in infinite time.

(c) The system is approximately observable in infinite time.

(d) The semigroup generator $\mathcal{A}_{\mathcal{L}}$ *does not have eigenvalues on* $i\,\mathbb{R}$.

(e) $\mathbb{T}$ *is weakly stable (equivalently,* $\mathbb{T}^*$ *is weakly stable).*

(f) $\mathbb{T}^*$ *is asymptotically stable.*

(g) The system is exactly controllable in infinite time.

(h) The system is approximately controllable in infinite time.

(i) $\mathcal{A}_{\mathcal{L}}^*$ *does not have eigenvalues on* $i\,\mathbb{R}$.

PROOF: (a) ⇒ (b): If $\mathbb{T}$ is asymptotically stable, then from (3.10) we obtain

$$\lim_{t_f \to \infty} \int_0^{t_f} \|y(\tau)\|_Y^2 \, d\tau = \frac{1}{2} \|x_0\|_X^2 \qquad \forall\, x_0 \in D(\mathcal{A}_{\mathcal{L}}).$$

The result ensues from this.

(b) ⇒ (c): The implication is obvious.

(c) ⇒ (d): Assume that $\mathcal{A}_{\mathcal{L}}$ has an eigenvalue λ on $i\,\mathbb{R}$ with corresponding eigenfunction x_0. Since the system is scattering energy preserving we must have (for $x_0 \in D(\mathcal{A}_{\mathcal{L}})$ and $u = 0$), see Remark 2.15,

$$\mathrm{Re}\, \langle \mathcal{A}_{\mathcal{L}} x_0, x_0 \rangle_X = - \|y\|_{\mathbb{R}}^2 \qquad \Rightarrow \qquad \mathrm{Re}\, \lambda \, \|x_0\|_X^2 = - \|y\|_{\mathbb{R}}^2 \, .$$

Thus if $\mathrm{Re}\, \lambda = 0$ it follows that $y = 0$ and the approximate observability gives $x_0 = 0$, which is a contradiction since x_0 is an eigenvector.

(a) ⇒ (e): The implication is obvious.

(d) ⇒ (a): (d) together with the compactness of the resolvent operator (see Theorem 2.28) as well as the contraction property of the semigroup show that the conditions on the famous stability theorem of Arendt and Batty [AB88] are satisfied. According to this theorem, the system is asymptotically stable.

(e) ⇒ (d): Since $\mathbb{T}$ is weakly stable, $\mathcal{A}_{\mathcal{L}}$ has no eigenvalues on $i\,\mathbb{R}$.

(e) $\iff$ (f) $\iff$ (g) $\iff$ (h) $\iff$ (i): Since the system is conservative, see Theorem 3.14, the proof is similar to the equivalence of (a)–(e), but with the dual system (see Remark 3.15) instead of the original system. ∎

3.4. Output energy preserving systems

Another supply rate that appears often in applications is the class of (strictly) output passive systems. This class appears mainly when static output feedback is used on an impedance passive system, see Section 5.1.1. We show that this class of systems is also conservative and well-posed in the sense of Weiss.

Definition 3.18. Let $\alpha \in \mathcal{L}(U)$ be such that it is *positive semi-definite*, i.e., $\alpha \geq 0$, and $\alpha \neq 0$. A system node $\mathcal{S}$ on (the real spaces) (U, X, U) is *output passive* if, for all $t > 0$, the solution (x, y) in Lemma 2.31 satisfies

$$\|x(t)\|_X^2 - \|x_0\|_X^2 \leq 2\int_0^t \langle u(\tau), y(\tau)\rangle_U \, d\tau - 2\int_0^t \langle \alpha y(\tau), y(\tau)\rangle_U \, d\tau. \tag{3.11}$$

The system is *strictly output passive* if α is coercive, i.e., $\langle \alpha y, y\rangle_U > \varepsilon \|y\|_U^2$. It is (*strictly*) *output energy preserving* if the above inequality hold in the form of an equality. Finally, $\mathcal{S}$ is (*strictly*) *output conservative* if both $\mathcal{S}$ and the dual system node $\mathcal{S}^*$ are (strictly) output energy preserving. ♣

Now consider a system as described in Theorem 2.14 which is output passive. Note that equation (3.11), in the energy preserving case, is equivalent to

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \left\|\alpha^{1/2} y(t)\right\|_U^2.$$

From (2.43)–(2.44) we conclude that

$$P_{W,\tilde{W}} = \begin{bmatrix} 0 & I \\ I & -2\alpha \end{bmatrix}.$$

This in turn implies, from (2.43)–(2.44), that

$$P_{W,\tilde{W}} = \begin{bmatrix} 0 & I \\ I & -2\alpha \end{bmatrix} = \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^{-T} \Sigma \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^{-1}. \tag{3.12}$$

From this follows that the matrices W and $\widetilde{W}$ in Theorem 2.14 satisfy

$$W^T \widetilde{W} + \widetilde{W}^T (W - 2\alpha \widetilde{W}) = \Sigma. \tag{3.13}$$

This helps to prove the following result.

Theorem 3.19: *Given a system node* $\mathcal{S} = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *as described in Theorem* 2.37*, which is (strictly) output energy preserving, i.e.,* $\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(y)\rangle_U$ *holds (see Theorem* 2.14*). Then, the adjoint system node (see Remark* 3.15*) is described by*

$$A\&B^* \begin{bmatrix} v \\ w \end{bmatrix} = -\mathcal{J}\mathcal{L}v$$

$$D(\mathcal{S}^*) = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \,\middle|\, \mathcal{L}v \in H^N(a,b;\mathbb{R}^n),\ (2\alpha\widetilde{W} - W)\begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} = w \right\},$$

$$C\&D^* \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix}.$$

Furthermore, the system is also (strictly) output conservative. If, in addition, α *is coercive, i.e.,* $\langle \alpha y, y\rangle_U \geq \varepsilon^2 \|y\|_U$ *for* $\varepsilon > 0$*, then the system is well-posed in the sense of Weiss.*

Remark 3.20. Here, for simplicity, we assume that α is symmetric. In the case α being not symmetric it is just enough to replace the term 2α by $\alpha^T + \alpha$ in all expressions. ♣

Remark 3.21. Note that the systems described in Theorem 3.11 correspond to the $\alpha = 0$ case when compared to the systems described in Theorem 3.19. However, the adjoint systems found in those theorems differ in a sign since the outputs also differ in a sign. ♣

PROOF (PROOF OF THEOREM 3.19): First we find the adjoint of the system node as follows. Let $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S})$ and $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$. Following the same procedure used in the proof of Theorem 3.11 one can show $\mathcal{L}v \in H^N(a,b)^n$. Thus, we can substitute (2.35) in equation (3.9), which gives

$$\left\langle \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} = -\langle x, \mathcal{J}\mathcal{L}v\rangle_X + \left\langle \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, \Sigma \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U + \left\langle \widetilde{W}\begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, w \right\rangle_U,$$

and using equation (3.13) yields

$$= -\langle x, \mathcal{J}\mathcal{L}v\rangle_X + \left\langle \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, (W^T\widetilde{W} + \widetilde{W}^T(W - 2\alpha\widetilde{W}))\begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U + \left\langle \widetilde{W}\begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, w \right\rangle_U.$$

Since $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(\mathcal{S})$ we obtain from Theorem 2.37 and the equation above

$$\begin{aligned}
\left\langle \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} &= -\langle x, \mathcal{J}\mathcal{L}v\rangle_X + \left\langle u, \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U \\
&+ \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, (W - 2\alpha\widetilde{W}) \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U + \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, w \right\rangle_U \\
&= \langle x, -\mathcal{J}\mathcal{L}v\rangle_X + \left\langle u, \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} \right\rangle_U \\
&+ \left\langle \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, (W - 2\alpha\widetilde{W}) \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} + w \right\rangle_U .
\end{aligned}$$

Again, as it was done at the end of the proof of theorem 3.11 we can conclude from this, together with the surjectivity of the boundary operator, and the definition of the adjoint operator that

$$(A\&B)^* = -\mathcal{J}\mathcal{L}, \quad (C\&D)^* \begin{bmatrix} v \\ w \end{bmatrix} = \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix}$$

and that

$$D(\mathcal{S}^*) = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \middle| \mathcal{L}v \in H^N(a,b)^n \text{ and } (2\alpha\widetilde{W} - W) \begin{bmatrix} f_{\partial,\mathcal{L}v} \\ e_{\partial,\mathcal{L}v} \end{bmatrix} = w \right\}.$$

From this (see Theorem 2.37) we clearly obtain the desired result. Next we prove that the system is conservative, i.e., that the dual system is also output energy preserving. This follows from

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v(t)\|_{\mathcal{L}}^2 = \frac{1}{2}\frac{d}{dt}\langle v(t), v(t)\rangle_{\mathcal{L}} &= -\langle \mathcal{J}\mathcal{L}v(t), v(t)\rangle_{\mathcal{L}} \\
&= -\frac{1}{2}\left\langle \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix}, \Sigma \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix} \right\rangle_U \\
&= -\frac{1}{2}\left\langle \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix}, (W^T\widetilde{W} + \widetilde{W}^T(W - 2\alpha\widetilde{W})) \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix} \right\rangle_U \\
&= \frac{1}{2}\left\langle w(t) - 2\alpha\widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix}, \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}v}(t) \\ e_{\partial,\mathcal{L}v}(t) \end{bmatrix} \right\rangle_U + \frac{1}{2}\tilde{y}(t)^T w(t) \\
&= \tilde{y}(t)^T w(t) - \left\langle \alpha\tilde{y}(t), \tilde{y(t)} \right\rangle_U ,
\end{aligned}$$

where we used (2.35) and (3.13). It only remains to prove the well-posedness. Integrating $\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_U$ from 0 to t_f and using the fact that α is coercive, i.e., $\langle \alpha y, y \rangle_U \geq \varepsilon^2 \|y\|_U$ for $\varepsilon > 0$, we find that

$$
\begin{aligned}
\|x(t_f)\|_{\mathcal{L}}^2 - \|x(0)\|_{\mathcal{L}}^2 =\ & 2\int_0^{t_f} u^T(\tau)y(t)\, d\tau - 2\int_0^{t_f} y(\tau)^T \alpha y(\tau) d\tau \\
& \leq 2\int_0^{t_f} u^T(\tau)y(t)\, d\tau - 2\varepsilon^2 \int_0^{t_f} \|y(\tau)\|_U^2\, d\tau \\
& \leq \frac{1}{\varepsilon^2}\int_0^{t_f} \|u(\tau)\|_U^2\, d\tau + \varepsilon^2 \int_0^{t_f} \|y(\tau)\|_U^2\, d\tau - 2\varepsilon^2 \int_0^{t_f} \|y(\tau)\|_U^2\, d\tau \\
& = \frac{1}{\varepsilon^2}\int_0^{t_f} \|u(\tau)\|_U^2\, d\tau - \varepsilon^2 \int_0^{t_f} \|y(\tau)\|_U^2\, d\tau,
\end{aligned}
$$

where we used $2a^T b \leq (a^T a)/\varepsilon^2 + \varepsilon^2(b^T b)$ for any $a, b \in \mathbb{R}^n$. From the inequality above, the result follows, see Definition 3.1. ■

Following this, it is easy to see that similar results to those of Theorems 3.16 and 3.17 also hold for this class of systems. That is, we can link exponential stability, observability and controllability of the class of output energy preserving systems described in the previous theorem. This is shown in the following theorems, whose proof follows the same ideas of the proof of Theorems 3.16 and 3.17.

Theorem 3.22: *Consider a BCS as described in Theorem* 2.14*. Assume it is strictly output energy preserving, i.e.,* $\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_U$ *holds. Furthermore, assume that* α *coercive. Then the following statements are equivalent:*

(a) The system is exactly controllable in time t_f*, hence for all* $t > t_f$*.*

(b) The system is exactly observable in time t_f*, hence for all* $t > t_f$*.*

(c) The semigroup $\mathbb{T}$ *generated by* $\mathcal{A}_{\mathcal{L}}$ *satisfies* $\|\mathbb{T}(t_f)\| < 1$ *(in particular, the system is exponentially stable).*

PROOF: The proof follows the same ideas of the proof of Theorem 3.16 with minor modifications. ■

Remark 3.23. In the case of the input space U being finite-dimensional, the condition on α being coercive is equivalent to the system being *strictly* output energy preserving. This follows since the condition $\alpha > 0$ in a finite-dimensional space necessarily implies that the inverse is bounded. ♣

Theorem 3.24: *Consider a BCS as described in Theorem 2.14. Assume it is output energy preserving, i.e.,* $\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_U$ *holds with* α *positive semi-definite, i.e.,* $\alpha \geq 0$*. If the semigroup* $\mathbb{T}$ *generated by* $\mathcal{A}_{\mathcal{L}}$ *satisfies* $\|\mathbb{T}(t_f)\| < 1$ *(in particular, the system is exponentially stable), then*

- *the system is exactly controllable in time* t_f *and*
- *the system is exactly observable in time* t_f*.*

PROOF: Since the system is output energy preserving it satisfies for all $x_0 \in D(\mathcal{A}_{\mathcal{L}})$ and $u(t) = 0$

$$\|x(t_f)\|_X^2 + 2\int_0^{t_f} \langle \alpha\, y(\tau), y(\tau)\rangle_Y \; d\tau = \|x_0\|_X^2 \tag{3.14}$$

$$\Rightarrow \quad \|x_0\|_X^2 \leq \|x(t_f)\|_X^2 + k_1 \int_0^{t_f} \|y(\tau)\|_Y^2 \; d\tau$$

for a $k_1 > 0$. By assumption we have $\|\mathbb{T}(t_f)\| = (1 - k_T^2)$ for some $k_T > 0$, and hence $\|x(t_f)\|_X = \|\mathbb{T}(t_f)x_0\|_X \leq (1 - k_T^2)\,\|x_0\|_X$. This together with (3.14) and the boundedness of α implies that $\int_0^{t_f} \|y(\tau)\|_Y^2 \; d\tau \geq k_2 \|x_0\|_X^2$ for some $k_2 > 0$. The exact observability now follows from Definition 3.7.

The assumption on $T(t)$ is equivalent to $\|\mathbb{T}^*\| < 1$. Recall that exact controllability is equivalent to the dual system being exactly observable in time t_f. According to Theorem 3.19 the dual system (see Remark 3.15) is also output energy preserving, thus, the result now follows from the exact observability proved above.■

Theorem 3.25: *Consider a BCS as described in Theorem 2.14. Assume it is strictly output energy preserving, i.e.,* $\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_U$ *holds with* α *coercive. Then the following statements are equivalent:*

(1) $\mathbb{T}$ *is asymptotically stable.*

(2) The system is exactly observable in infinite time.

(3) The system is approximately observable in infinite time.

(4) The semigroup generator $\mathcal{A}_{\mathcal{L}}$ *does not have eigenvalues on* $i\,\mathbb{R}$*.*

(5) $\mathbb{T}$ *is weakly stable (equivalently,* $\mathbb{T}^*$ *is weakly stable).*

(6) $\mathbb{T}^*$ *is asymptotically stable.*

(7) The system is exactly controllable in infinite time.

(8) The system is approximately controllable in infinite time.

(9) $\mathcal{A}_{\mathcal{L}}^*$ does not have eigenvalues on $i\,\mathbb{R}$.

PROOF: The proof follows the same ideas of the proof of Theorem 3.17 with minor modifications (noticing that the conservativity of the system follows from Theorem 3.19). ∎

Theorem 3.26: *Consider a BCS as described in Theorem 2.14. Assume it is output energy preserving, i.e., $\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_U$ with α positive semi-definite. If the semigroup generator $\mathcal{A}_{\mathcal{L}}$ does not have eigenvalues on $i\,\mathbb{R}$ then*

- *$\mathbb{T}$ is asymptotically stable,*
- *the system is exactly observable in infinite time,*
- *the system is approximately observable in infinite time,*
- *$\mathbb{T}$ is weakly stable (equivalently, $\mathbb{T}^*$ is weakly stable),*
- *$\mathcal{A}_{\mathcal{L}}^*$ does not have eigenvalues on $i\,\mathbb{R}$,*
- *$\mathbb{T}^*$ is asymptotically stable,*
- *the system is exactly controllable in infinite time, and*
- *the system is approximately controllable in infinite time.*

PROOF: The assumption on $\mathcal{A}_{\mathcal{L}}$ together with the compactness of the resolvent operator (see Theorem 2.28) as well as the contraction property of the semigroup show that the conditions on the famous stability theorem of Arendt and Batty [AB88] are satisfied. According to this theorem, the system is asymptotically stable (and hence weakly stable).

Since the system is asymptotically stable it follows from (3.14) and the boundedness of α that

$$\lim_{t_f\to\infty}\int_0^{t_f} \|y(\tau)\|_Y^2 \, d\tau \geq \frac{1}{k_1}\|x_0\|_X^2 \qquad \forall\, x_0 \in D(\mathcal{A}_{\mathcal{L}}).$$

The observability ensues from this.

Since $\mathbb{T}$ is weakly stable we also have that $\mathbb{T}^*$ is weakly stable. This in turn implies that $\mathcal{A}_{\mathcal{L}}^*$ does not have eigenvalues on $i\,\mathbb{R}$. Since the system is conservative, see Theorem 3.19, the rest of the proof follows the same ideas as above, but with the dual system (see Remark 3.15) instead of the original system. ■

Example 3.27 As an example of an output energy preserving we consider the Timoshenko beam studied in Example 2.19. Recall that the model is described by (2.53), the energy is given by (2.54) and the ports by (2.55). In this case we impose the following boundary conditions

$$\begin{aligned} &\frac{1}{\rho(a)}\,x_2(a,t)=0, \quad \frac{1}{I_\rho(a)}\,x_4(a,t)=0, \quad t\geq 0,\\ &K(b)\,x_1(b,t) = -\alpha_1\,\frac{1}{\rho(b)}\,x_2(b,t), \quad EI(b)\,x_3(b,t) = -\alpha_2\,\frac{1}{I_\rho(b)}\,x_4(b,t), \end{aligned} \tag{3.15}$$

where α_1 and α_2 are given positive gain feedback constants. These conditions correspond to a beam clamped at the left side, i.e., at $z=a$, and controlled at $z=b$ by force and moment feedback. The corresponding matrix W that gives the boundary conditions above is

$$W=\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1\\ \alpha_1 & 1 & 0 & 0 & 1 & \alpha_1 & 0 & 0\\ 0 & 0 & \alpha_2 & 1 & 0 & 0 & 1 & \alpha_2 \end{bmatrix}, \tag{3.16}$$

which satisfies

$$W\Sigma W^T = 2\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \alpha_1 & 0\\ 0 & 0 & 0 & \alpha_2 \end{bmatrix}.$$

As output we choose

$$y=\begin{bmatrix} -K(a)x_1(a)\\ -(EI)(a)x_3(a)\\ \frac{1}{\rho(b)}\,x_2(b,t)\\ \frac{1}{I_\rho(b)}\,x_4(b,t) \end{bmatrix}, \quad \text{with} \quad \widetilde{W}=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, from equation (2.30) we obtain again that

$$P_{W,\tilde{W}}^{-1}=\begin{bmatrix} 2\alpha & I\\ I & 0 \end{bmatrix}, \quad P_{W,\tilde{W}}=\begin{bmatrix} 0 & I\\ I & -2\alpha \end{bmatrix}, \quad \alpha=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \alpha_1 & 0\\ 0 & 0 & 0 & \alpha_2 \end{bmatrix},$$

which, in turn gives from (2.29) that

$$\frac{d}{dt}E(t)=\frac{1}{2}\frac{d}{dt}\left\|x(t)\right\|_{\mathcal{L}}^2=\langle u(t),y(t)\rangle_U-\langle \alpha y(t),y(t)\rangle_U\,.$$

Observe that this system corresponds to the closed-loop system of an impedance energy preserving system with a static controller, see Figure 3.1. Indeed if we let $\alpha = 0$ in the equation above, we obtain an impedance energy preserving system with input $u = \left(\frac{1}{\rho(a)}\, x_2(a,t),\ \frac{1}{I_\rho(a)}\, x_4(a,t),\ K(b)\, x_1(b,t),\ EI(b)\, x_3(b,t)\right)^T$ and output $y = \left(-K(a)x_1(a),\ -(EI)(a)x_3(a),\ \frac{1}{\rho(b)}\, x_2(b,t)\,,\ \frac{1}{I_\rho(b)}\, x_4(b,t)\right)^T$.

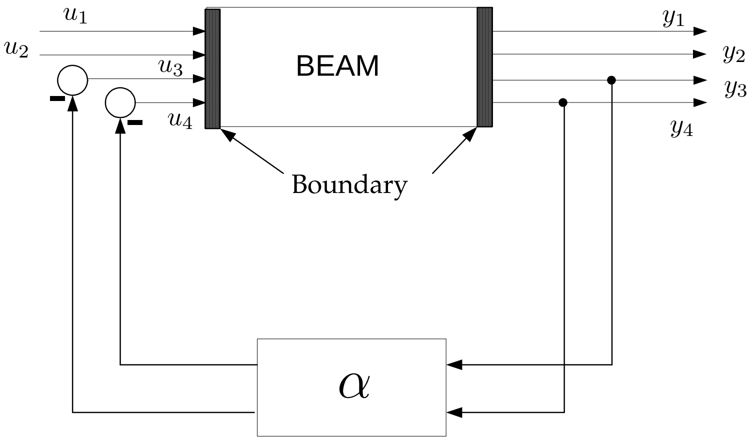

Figure 3.1.: Feedback loop.

It is easy to prove that the system with the boundary conditions (3.15) (the closed-loop system) is asymptotically stable. For instance, this can be done by proving that the semigroup generator $\mathcal{A}_\mathcal{L}$ does not have eigenvalues in the imaginary axis, see Theorem 3.26. In fact, assume that λ is an eigenvalue of $\mathcal{A}_\mathcal{L}$ with corresponding eigenfunction $\tilde{x}$. Then from the equation above we obtain (see Remark 2.15)

$$\langle \mathcal{A}_\mathcal{L}\tilde{x}, \tilde{x}\rangle_\mathcal{L} = \operatorname{Re}\lambda\, \langle \tilde{x}, \tilde{x}\rangle_\mathcal{L} = -\, \langle \alpha y(t), y(t)\rangle_U\, .$$

If $\operatorname{Re}\lambda = 0$ then $\langle \alpha y(t), y(t)\rangle_U$ must be zero, which in turn implies that $\frac{1}{\rho(b)}\, \tilde{x}_2(b,t)$ and $\frac{1}{I_\rho(b)}\, \tilde{x}_4(b,t)$ are zero since α_1 and α_2 are nonzero constants. Using this in the boundary conditions (3.15) gives that $K(b)\, \tilde{x}_1(b,t)$ and $EI(b)\, \tilde{x}_3(b,t)$ are zero too. Thus, $\tilde{x}$ is the solution of a PDE with *all* boundary variables at $z = b$ set to zero for all $t \geq 0$. Therefore, we can conclude from Holmgren's theorem, see [Joh49], that $\tilde{x} = 0$, which is a contradiction since $\tilde{x}$ was assumed to be an eigenvector. Hence $\mathcal{A}_\mathcal{L}$ does not have eigenvalues on the imaginary axis. Since α is clearly positive semi-definite (not coercive) we can apply Theorem 3.26. It is also possible to prove that this system is exponentially stable, but the proof is more involved, see Example 4.30 for details. $*$

Chapter 4

Riesz Basis Property: Case $\mathrm{N} = 1$

In this chapter we investigate the Riesz basis property of the class of boundary control systems described by (2.1)–(2.3) with $N = 1$. This class appears often in the literature and covers many interesting applications such as the control of vibrations in flexible structures and traveling waves in acoustics. The property of the generalized eigenvectors of the system forming a Riesz basis is one of the most important features in the analysis of distributed parameter systems both from a theoretical and practical point of view. The validity of this property results not only in the fact that the stability of the system is determined by the spectrum of the semigroup generator, which is referred to as the spectrum-determined growth condition, but also is important since the dynamic behavior of the system can be described in the form of eigenfunction expansions of non-harmonic Fourier series. Unfortunately, this property is not easy to verify, even for the most studied systems such as Timoshenko and Euler-Bernoulli beams under certain linear boundary feedback control. Many of the methods used to verify the Riesz basis property rely heavily on the exact expression of the characteristic equation and the eigenfunctions, which make them not useful when dealing with variable coefficients. Also many of the methods used to check this conditions can only be applied in a case-by-case basis. On the other hand, if one is only interested in exponential stability, then one can avoid the Riesz basis approach by using the multiplier technique, see [Kom95], but this is also case-dependent due to the search of the suitable multiplier and in some cases it is not obvious how to select such multiplier. So it appears that not many results have been published that handle the Riesz basis property for a class of systems.

The main aim of this chapter is to deal with a class of systems rather than tackling a particular problem. To do so, we take advantage of the results presented in Chapter 2. Once we have posed the problem as described in that chapter we use the results presented in [MM03] and [Tre00b] to prove the Riesz basis property. It is well known that the study of systems with variable coefficient is difficult because explicit solution formula is hard to find. Instead, our approach only requires asymptotic (not necessarily exact) expressions of the characteristic equations, eigenvalues and eigenfunctions. These asymptotic expression emerge directly from the method described in [Tre00b].

Recently, Xu and Yung, [XY05], introduced a simple criterion to establish the Reisz basis property, which has already been used in [XY04], [Xu05]. The verification of this property in [XY05] is based on the assumption that the spectrum of the main operator (i.e., the semigroup generator) lies inside some vertical strip, its eigenvalues are separated, and the algebraic multiplicity of the eigenvalues is bounded uniformly. Under these conditions the eigenfunctions of the semigroup operator form a Riesz basis on the state space provided that the eigenfunctions are complete on that space. The main result of this chapter is that it proves the Riesz basis property for a class of systems only based on the assumption that the eigenvalues are separated. That is, we show for this class of systems, that the spectrum of the system lies inside some vertical strip, the algebraic multiplicity of the eigenvalues are uniformly bounded, and, mainly, that the eigenfunctions are complete in the state space. We remark that this class of systems contains the well-know Timoshenko beam and the wave equation, as well as models of heat exchangers ([KS98]), swelling porous elastic soils with fluid saturation ([WG06]), linear bioprocess model with recycle loop ([SK05]) among others. For more details on Riesz basis see [You80] or [CZ95b].

Recall that we denote by $M_{n,m}(G)$ the set of all $n \times m$ matrices with entries in G. If $n = m$ we write $M_n(G)$. For $m \in \mathbb{N}$ and $p \in \mathbb{R}$ with $1 \leq p \leq \infty$, we define the *Sobolev space*

$$W^{m,p}(a,b) = \left\{ v \in L_p(a,b) \mid \frac{\partial^\alpha}{\partial z^\alpha} v \in L_p(a,b),\ \forall \alpha \leq m,\ \alpha \in \mathbb{N} \right\}, \tag{4.1}$$

which is a Banach space for the norm

$$\|u\|_{m,p,(a,b)} = \left(\sum_{\alpha=0}^{m} \int_a^b \left| \frac{\partial^\alpha}{\partial z^\alpha} u(x) \right|^p dx \right)^{1/p}, \quad p < \infty \tag{4.2}$$

or

$$\|u\|_{m,\infty,(a,b)} = \sup_{|\alpha| \leq m} \left(\sup_{x \in (a,b)} \text{ess} \left| \frac{\partial^\alpha}{\partial z^\alpha} u(x) \right| \right), \quad p = \infty.$$

Here, the derivative is in the distributional sense. If $p = 2$ we write $H^m(a,b)$ instead of $W^{m,2}(a,b)$. The standard norm and inner product on $L_2(a,b)$ (or $L_2(a,b)^n$) will be denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ respectively.

We start by briefly recalling the fundamental matrix of a differential equation and its properties which will be used in the sequel. Then we study the asymptotic properties of the BCS. Finally, we proceed to prove the main results. But first, let us recall the definition of a Riesz basis.

A sequence $\{y_i\}$ in a Hilbert space H is called a *basis* for H if to each element $x \in H$ corresponds a unique sequence of scalars c_i, $i = 1, 2, \ldots, K$ such that the series

$$x = \sum_{i=1}^{K} c_i\, y_i \tag{4.3}$$

is convergent with respect to the norm of H. $\{y_i\}$ is called a *Riesz basis* for H if

i) $\overline{\text{span}}\{y_i\} = H$, and

ii) There exist positive constants m and M such that for an arbitrary positive integer K and arbitrary scalars c_i, $i = 1, 2, \ldots, K$ one has

$$m \sum_{i=1}^{K} |c_i|^2 \leq \left\| \sum_{i=1}^{K} c_i y_i \right\|_H^2 \leq M \sum_{i=1}^{K} |c_i|^2.$$

A basis $\{y_i\}$ of H is called a *Riesz basis with parentheses*, see [Shk86], if (4.3) converges in H after putting some of its terms in parentheses the arrangement of which does not depend on x. Refer to [You80] for more details on Riesz bases.

4.1. The fundamental matrix

Definition 4.1. Suppose $P(z) \in M_n(L_p(a,b))$ and let

$$v'(z) - P(z)v(z) = 0 \qquad z \in [a,b]. \tag{4.4}$$

Then a matrix $M(z) \in M_n(W^{1,p}(a,b))$ is called a *fundamental matrix* for the equation above if for each independent solution of (4.4), say v, there exists a $c \in \mathbb{R}^n$ such that $v(z) = M(z)c$. Equivalently, $M(z)$ is invertible and $M'(z) = P(z)M(z)$. If in addition, $M(z)$ satisfies $M(a) = I$, then it is unique and is called the *transition matrix*. ♣

Note that $M(z)\,C$ is also a fundamental matrix if $C \in M_n(\mathbb{R})$ is any nonsingular matrix, since $M(z)\,C$ is also nonsingular and its columns are linear combinations of $M(z)$.

It is well-known that if $P(z)$ satisfies (the Lappo-Danilevskii condition)

$$P(z) \int_a^z P(\tau)\, d\tau = \int_a^z P(\tau)\, d\tau\, P(z),$$

then $M(z) = \exp\left(\int_a^z P(\tau)\, d\tau\right)$. In particular, if $P(z) = P$ is a constant matrix, then $M(z) = \mathrm{e}^{P(z-a)}$.

The transition matrix makes it possible to write every solution of the inhomogeneous system

$$v'(z) = P(z)v(z) + w(z) \qquad z \in [a, b]$$

in the form of Cauchy's formula

$$v(z) = M(z)v(a) + \int_a^z M(z, \tau)w(\tau)\, d\tau \qquad z \in [a, b], \tag{4.5}$$

where

$$M(z, \tau) = M(z)\, M^{-1}(\tau) \tag{4.6}$$

is called the *Cauchy matrix* of (4.4). The Cauchy matrix is jointly continuous in its arguments on $[a, b] \times [a, b]$ and for arbitrary $z, \tau \in [a, b]$ it has the properties

$$M(z, \tau) = M(z, a)\, M^{-1}(\tau, a), \tag{4.7a}$$

$$M(z, \tau) = M(z, s)\, M(s, \tau), \tag{4.7b}$$

$$M(z, \tau) = M^{-1}(\tau, z), \tag{4.7c}$$

$$M(z, z) = I, \tag{4.7d}$$

if $H(z, \tau)$ is the Cauchy matrix of the adjoint system $v'(z) = P^T(z)v(z)$, then $H(z, \tau) = M^T(z, \tau)$. (4.7e)

4.2. Case $N = 1$ with variable coefficients

Here we consider systems of the form (see Theorem 2.14)

$$\begin{aligned} \tfrac{\partial x}{\partial t}(t, z) &= P_1 \tfrac{\partial \mathcal{L}x(t,z)}{\partial z} + P_0 \mathcal{L}x(t, z), \quad z \in [a, b], \quad x(0) = x_0, \\ u(t) &= W \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}, \\ y(t) &= C \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}, \end{aligned} \tag{4.8}$$

where $\mathcal{L}$ is a coercive operator on $L_2(a, b)^n$, i.e., $\mathcal{L} = \mathcal{L}^*$ and $\langle x, \mathcal{L}x \rangle \geq \delta \|x\| > 0$; $P_0 = -P_0^T \in M_n(\mathbb{R})$, $P_1 = P_1^T \in M_n(\mathbb{R})$, and

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix}, \tag{4.9}$$

see Definition 2.5. Note that $\mathcal{L}$ can depend on z, i.e., the PDE above can have variable coefficients. In this chapter we assume $\mathcal{L} \in M_n(W^{2,\infty}(a, b))$ and we consider $\mathcal{L}$ as a multiplication operator on $L_2(a, b)^n$. The state space is given by (2.33), i.e.,

$$X = L_2(a, b)^n \text{ with the inner product } \langle \cdot, \cdot \rangle_{\mathcal{L}} = \langle x, \mathcal{L}x \rangle .$$

If $W = \begin{bmatrix} W_1, & W_2 \end{bmatrix}$ satisfies $W \Sigma W^T \geq 0$, then we have that the operator $\mathcal{A}_\mathcal{L}$ defined by

$$\mathcal{A}_\mathcal{L} x = P_0 \mathcal{L} x + P_1 \frac{\partial \mathcal{L} x}{\partial z}, \tag{4.10a}$$

with domain

$$\begin{aligned} D(\mathcal{A}_\mathcal{L}) = & \left\{ \mathcal{L}x \in H^1(a,b;\mathbb{R}^n) \mid \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} \in \ker W \right\}, \\ = & \left\{ \mathcal{L}x \in H^1(a,b;\mathbb{R}^n) \mid W_b(\mathcal{L}x)(b) + W_a(\mathcal{L}x)(a) = 0 \right\} \end{aligned} \tag{4.10b}$$

generates a contraction semigroup, where

$$W_b := (W_1 P_1 + W_2), \quad W_a := (-W_1 P_1 + W_2). \tag{4.10c}$$

First we study the eigenvalues of $\mathcal{A}_\mathcal{L}$. Since $\mathcal{A}_\mathcal{L}$ generates a contraction semigroup we know from the Lümer-Phillips theorem that Re $\langle \mathcal{A}_\mathcal{L} x, x \rangle_\mathcal{L} \leq 0$ for all $x \in D(\mathcal{A}_\mathcal{L})$. Let x_1 be any eigenvector of $\mathcal{A}_\mathcal{L}$ with corresponding eigenvalue λ. Then we have that

$$0 \geq \text{Re } \langle \mathcal{A}_\mathcal{L} x_1, x_1 \rangle_\mathcal{L} = \text{Re } \langle \mathcal{A}_\mathcal{L} x_1, \mathcal{L} x_1 \rangle = \text{Re } \langle \lambda x_1, \mathcal{L} x_1 \rangle = \text{Re}\, \lambda \, \langle x_1, \mathcal{L} x_1 \rangle .$$

This implies (by the coercivity of $\mathcal{L}$) that all eigenvalues of $\mathcal{A}_\mathcal{L}$ satisfy Re $\lambda \leq 0$. Moreover, we have that this eigenvector x_1 is the solution of

$$P_0 \mathcal{L}\, x_1(z) + P_1 \frac{d\mathcal{L}x_1}{dz}(z) = \lambda x_1(z) \iff \frac{d\mathcal{L}x_1}{dz}(z) = P_1^{-1}(\lambda \mathcal{L}^{-1}(z) - P_0)\mathcal{L}x_1(z). \tag{4.11}$$

The general solution of the equation above is given by

$$\mathcal{L}x_1(z) = M(z,\lambda)\, c, \qquad M(a,\lambda) = I \tag{4.12}$$

where $M(z,\lambda)$ is a transition matrix of (4.11), and c is a constant vector determined by the boundary conditions. If $\mathcal{L}$ is a constant matrix, then we can write $M(z,\lambda) = \mathrm{e}^{P_1^{-1}(\lambda \mathcal{L}^{-1} - P_0)(z-a)}$. Note the dependance on the eigenvalue parameter λ of this transition matrix. Do not confuse with the Cauchy matrix.

Using the boundary conditions on $D(\mathcal{A}_\mathcal{L})$, see (4.10b), we conclude that the solution of (4.11) has to satisfy

$$W_b(\mathcal{L}x_1)(b) + W_a(\mathcal{L}x_1)(a) = 0. \tag{4.13}$$

Using (4.12) in the above equation gives

$$(W_b\, M(b,\lambda) + W_a)\, c = 0. \tag{4.14}$$

We know that λ is an eigenvalue of $\mathcal{A}_\mathcal{L}$ if and only if the matrix above is singular. We showed above that the eigenvalues of $\mathcal{A}_\mathcal{L}$ satisfy $\operatorname{Re}\lambda \le 0$, which shows that the matrix above is nonsingular if $\operatorname{Re}\lambda > 0$, otherwise $\mathcal{A}_\mathcal{L}$ would have eigenvalues with $\operatorname{Re}\lambda > 0$. In resume we have the following result.

Theorem 4.2: *Consider the operator $\mathcal{A}_\mathcal{L}$ described in* (4.10) *with $\mathcal{L}$ a multiplication operator and $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ satisfying $W\Sigma W^T \ge 0$. Denote the transition matrix of* (4.11) *by $M(z, \lambda)$, see* (4.12) *and by $M(z, s, \lambda)$ the respective Cauchy matrix. Then,*

(i) The eigenvalues of $\mathcal{A}_\mathcal{L}$ satisfy $\operatorname{Re}\lambda \le 0$, and are the λ's for which the matrix

$$H_0(\lambda) = W_b\, M(b, \lambda) + W_a \tag{4.15}$$

is singular. If $\operatorname{Re}\lambda > 0$, the matrix above is nonsingular.

(ii) The corresponding eigenfunction is given by

$$x_i(z) = \mathcal{L}^{-1}(z) M(z, \lambda_i) c, \tag{4.16}$$

where c is a nonzero solution of $H_0(\lambda_i)c = 0$.

(iii) The resolvent $(\lambda - \mathcal{A}_\mathcal{L})^{-1}$ can be represented as

$$\left((\lambda - \mathcal{A}_\mathcal{L})^{-1} y\right)(z) = \mathcal{L}^{-1}\left(M(z, \lambda)\, c - \int_a^z M(z, \tau, \lambda)\, P_1^{-1} y(\tau)\, d\tau\right), \tag{4.17}$$

where c is

$$c = H_0^{-1}(\lambda)\, W_b \int_a^b M(b, \tau, \lambda)\, P_1^{-1} y(\tau)\, d\tau. \tag{4.18}$$

(iv) Furthermore, if $\lambda > 0$, the resolvent operator $(\lambda - \mathcal{A}_\mathcal{L})^{-1}$ is a compact operator, and consequently the spectrum of $\mathcal{A}_\mathcal{L}$, $\sigma(\mathcal{A}_\mathcal{L})$, consists only of isolated eigenvalues with finite multiplicity. That is, $\sigma(\mathcal{A}_\mathcal{L}) = \sigma_p(\mathcal{A}_\mathcal{L}) = \{\lambda \mid \det H_0(\lambda) = 0\}$.

PROOF: Points (*i*) and (*ii*) were proved above. Next we study the resolvent of $\mathcal{A}_\mathcal{L}$. First we study the range of $(\lambda - \mathcal{A}_\mathcal{L}) : D(\mathcal{A}_\mathcal{L}) \to X$ for $\lambda > 0$. To see this, consider $(\lambda - \mathcal{A}_\mathcal{L})\, x = y$, which is equivalent to solve

$$\begin{aligned} y = (\lambda - \mathcal{A}_\mathcal{L})\, x \quad &\Longleftrightarrow \quad y = \lambda\, x - P_0 \mathcal{L}\, x - P_1 \frac{d\mathcal{L}x}{dz} \\ \Longleftrightarrow \frac{d\mathcal{L}x}{dz}(z) &= P_1^{-1}(\lambda \mathcal{L}^{-1} - P_0)\, \mathcal{L}x(z) - P_1^{-1}\, y(z) \end{aligned} \tag{4.19}$$

for $x \in D(\mathcal{A}_\mathcal{L})$. The general solution of (4.19) is given by, see (4.5),

$$\mathcal{L}x(z) = M(z,\lambda)\, c - \int_a^z M(z,\tau,\lambda)\, P_1^{-1} y(\tau)\, d\tau, \tag{4.20}$$

where c is a constant vector determined by the boundary conditions. Since $x \in D(\mathcal{A}_\mathcal{L})$ the boundary conditions (see (4.10b)) are described by (4.13). Using (4.20) in (4.13) yields

$$\begin{aligned} W_b \left[M(b,\lambda)\, c - \int_a^b M(b,\tau,\lambda)\, P_1^{-1} y(\tau)\, d\tau \right] + W_a\, c = 0 \\ \iff H_0(\lambda)\, c = W_b \int_a^b M(b,\tau,\lambda)\, P_1^{-1} y(\tau)\, d\tau. \end{aligned}$$

We already showed that when $\operatorname{Re}\lambda > 0$ the matrix $H_0(\lambda)$ is nonsingular. In that case, c can be defined uniquely, which implies that (4.19) has a unique solution in $D(\mathcal{A}_\mathcal{L})$. That the resolvent is compact follows from Theorem 2.28. ■

Next, we study the asymptotic properties of the fundamental matrix above. We use the approach used in [Tre00b] and [MM86]. To do this we would like to diagonalize the matrix $(\mathcal{L}(z)P_1)^{-1}$. By using the coercivity of $\mathcal{L}$ we have that $\mathcal{L} = \mathcal{L}^{1/2}\mathcal{L}^{1/2}$ where $\mathcal{L}^{1/2}$ is also coercive. Using this we get, for any eigenvector $v(z)$ of $(\mathcal{L}(z)P_1)^{-1}$ with corresponding eigenvalue $\lambda(z)$, that

$$\begin{aligned} \lambda\, \langle v, u\rangle &= \langle P_1^{-1}\mathcal{L}^{-1} v, u\rangle = \left\langle P_1^{-1}\mathcal{L}^{-1/2}\mathcal{L}^{-1/2} v, \mathcal{L}^{-1/2}\mathcal{L}^{1/2} u\right\rangle \\ &= \left\langle \mathcal{L}^{-1/2}P_1^{-1}\mathcal{L}^{-1/2}(\mathcal{L}^{-1/2}v), \mathcal{L}^{1/2}u\right\rangle, \quad \forall u. \end{aligned}$$

But we also have that $\langle v, u\rangle = \langle \mathcal{L}^{-1/2}v, \mathcal{L}^{1/2}u\rangle$. Then we conclude (with $\widetilde{u} = \mathcal{L}^{1/2}u$)

$$\lambda \left\langle \mathcal{L}^{-1/2}v, \widetilde{u}\right\rangle = \left\langle \mathcal{L}^{-1/2}P_1^{-1}\mathcal{L}^{-1/2}(\mathcal{L}^{-1/2}v), \widetilde{u}\right\rangle, \quad \forall \widetilde{u}.$$

Hence $\lambda\,(\mathcal{L}^{-1/2}v) = \mathcal{L}^{-1/2}P_1^{-1}\mathcal{L}^{-1/2}(\mathcal{L}^{-1/2}v)$, which implies that λ is also an eigenvalue of a symmetric operator, and hence it is semisimple and a real function. Next we make the following assumption.

ASSUMPTION 4.3: Assume $\mathcal{L} \in M_n(W^{2,\infty}(a,b))$, P_1 is nonsingular, and the eigenvalues of $P_1^{-1}\mathcal{L}^{-1} \in M_n(W^{2,\infty}(a,b))$ satisfy

$$|\lambda_\mu - \lambda_\nu|^{-1} \in L_\infty(a,b).$$

This means that $\lambda_\mu(z)$ cannot coincide with some other $\lambda_\nu(x)$ at any point $x \in [a,b]$. ♡

The assumption above implies that the eigenvalues and eigenvectors of $P_1^{-1}\mathcal{L}^{-1}$ (as a function of z) are continuously differentiable, see [Kat95]. This implies that $P_1^{-1}\mathcal{L}^{-1}$ can be diagonlized. In other words, there exists an $R \in M_n(W^{2,\infty}(a,b))$ with $R^{-1} \in M_n(W^{2,\infty}(a,b))$, such that $R^{-1}(\mathcal{L}P_1)^{-1}R = A_1$ where $A_1(z) \in M_n(W^{2,\infty}(a,b))$ is a diagonal matrix with the eigenvalues of $P_1^{-1}\mathcal{L}^{-1}$ as diagonal elements, see [Kat95, Chapter II] or [Iva89].

Let $R(z)$ be a matrix that diagonalizes $(\mathcal{L}(z)P_1)^{-1}$, i.e., $R^{-1}(\mathcal{L}P_1)^{-1}R = A_1$ where $A_1(z) = \operatorname{diag}\{r_1(z),\ldots,r_n(z)\}$. Consider the eigenvalue problem (4.11) and let

$$\begin{aligned}\phi(z) = R^{-1}\mathcal{L}x_1(z), \quad A_1 = R^{-1}(\mathcal{L}P_1)^{-1}R = \operatorname{diag}\{r_1,\ldots,r_n\},\\ A_0 = -R^{-1}(P_1^{-1}P_0R + R'),\end{aligned} \tag{4.21}$$

where $R' = \frac{\partial}{\partial z}R$. Then, equation (4.11) becomes

$$\frac{d\phi}{dz}(z) = \big(\lambda A_1(z) + A_0(z)\big)\phi(z), \quad z \in [a,\, b]. \tag{4.22}$$

4.3. First order eigenvalue problem

In this section we look at the eigenvalue problem (4.21)–(4.22), i.e.,

$$\begin{aligned}&\frac{d\phi}{dz}(z) = (\lambda A_1(z) + A_0(z))\phi(z), \quad z \in (a,b) \\ &W_b(R\phi)(b) + W_a(R\phi)(a) = 0,\end{aligned} \tag{4.23}$$

where W_b and W_a are given by (4.10c). In this section we assume that $A_1,\, A_0 \in M_n(W^{0,\infty}(a,b)) = M_n(L_\infty(a,b))$, and A_1 has the diagonal form

$$A_1 = \begin{pmatrix} r_0 I_{n_0} & 0 & \ldots & 0 \\ 0 & r_1 I_{n_1} & & \vdots \\ \vdots & \ddots & & \\ 0 & \ldots & & r_l I_{n_l} \end{pmatrix}, \qquad |r_\nu - r_\mu|^{-1} \in L_\infty(a,b),\ \nu \neq \mu, \tag{4.24a}$$

$r_\nu(z) \in \mathbb{R}$ for all $z \in [a,b]$. The condition above comes from Assumption 4.3, from which we deduce that there are $\phi_{\nu\mu} \in \{0,\pi\}$ (since the eigenvalues of $\mathcal{L}\,P_1$ are real) such that

$$r_\nu(z) - r_\mu(z) = |r_\nu(z) - r_\mu(z)|\mathrm{e}^{i\phi_{\nu\mu}}, \quad \text{for } \nu,\mu \in \{0,\ldots,l\}. \tag{4.24b}$$

We set

$$
\begin{aligned}
R_\nu(z) &:= \int_a^z r_\nu(\xi)\,d\xi, \qquad \nu = 0,\dots,l;\ z \in [a,b],\\
E_\nu(z,\lambda) &:= \mathrm{e}^{\lambda R_\nu(z)} I_{n_\nu}, \qquad \nu = 0,\dots,l;\ z \in [a,b],\\
E(z,\lambda) &= \begin{pmatrix} E_0(z,\lambda) & 0 & \dots & 0 \\ 0 & E_1(z,\lambda) & & \vdots \\ \vdots & \ddots & & \\ 0 & \dots & & E_l(z,\lambda) \end{pmatrix}.
\end{aligned}
\tag{4.25}
$$

Let U be an unbounded subset of $\mathbb{C}$, f be a function on U with values in $M_{k,n}(\mathbb{C})$ and g be a complex-valued function on U. We write

$$f(\lambda) = O(g(\lambda))$$

if there is a $C > 0$ such that $|f(\lambda)| \le C|g(\lambda)|$ for $\lambda \in U$. The notation

$$f(\lambda) = o(g(\lambda))$$

means that $|f(\lambda)|\,|g(\lambda)|^{-1} \to 0$ as $|\lambda| \to \infty$ in U. Also, let $f(\cdot,\lambda) \in M_{k,n}(L_p(a,b))$ for $\lambda \in U$ and, as above, g be a complex-valued function on U. We write

$$f(\cdot,\lambda) = \{O(g(\lambda))\}_p \quad \text{or} \quad f(\cdot,\lambda) = O(g(\lambda)) \text{ in } M_{k,n}(L_p(a,b))$$

if there is a $C > 0$ such that $\|f(\cdot,\lambda)\|_p \le C|g(\lambda)|$ for $\lambda \in U$, and

$$f(\cdot,\lambda) = \{o(g(\lambda))\}_p \quad \text{or} \quad f(\cdot,\lambda) = o(g(\lambda)) \text{ in } M_{k,n}(L_p(a,b))$$

if $\|f(\cdot,\lambda)\|_p\,|g(\lambda)|^{-1} \to 0$ as $|\lambda| \to \infty$ in U.

For the matrices A_0, A_1 and Ψ_0 (defined below) we form the block matrices

$$A_j := (A_{j,\nu\mu})_{\nu,\mu=0}^{l}, \quad \text{and} \quad \Psi_0 := (\Psi_{0,\nu\mu})_{\nu,\mu=0}^{l}$$

according to the block structure of A_1. The following theorem can be found in either [MM03, §2.8], [MM86] or [Tre00b].

Theorem 4.4: *Let $0 \ne \lambda \in \mathbb{C}$. Consider the eigenvalue problem* (4.23)–(4.25). *Then, there exists a fundamental matrix $\widehat{\Psi}(z,\lambda)$ of* (4.23), *which satisfies*

$$\widehat{\Psi}'(z) = \big(\lambda A_1 + A_0\big)\widehat{\Psi}(z) \tag{4.26}$$

such that for large enough $|\lambda|$,

$$\widehat{\Psi}(z,\lambda) = \Big(\Psi_0(z) + \widehat{\Theta}(z,\lambda)\Big)\, E(z,\lambda), \tag{4.27}$$

where $\Psi_0(z) \in M_n(W^{1,\infty}(a,b))$ *is determined by*

$$\begin{aligned} &\Psi_0(z)A_1 = A_1\Psi_0(z), \qquad \Psi_0(a) = I, \\ &\Psi'_{0,\nu\nu} - A_{0,\nu\nu}\Psi_{0,\nu\nu} = 0, \quad \nu = 0,\dots,l \end{aligned} \tag{4.28}$$

and $\widehat{\Theta}(z,\lambda) \in M_n(W^{1,\infty}(a,b))$ *have, for large* λ*, the asymptotic estimates*

$$\begin{aligned} &\widehat{\Theta}(z,\lambda) = \{o(1)\}_\infty, \\ &\widehat{\Theta}(z,\lambda) = \{O(\tau_\infty(\lambda))\}_\infty, \\ &\frac{1}{\lambda}\widehat{\Theta}'(z,\lambda) = \{o(1)\}_\infty, \\ &\frac{1}{\lambda}\widehat{\Theta}'(z,\lambda) = \{O(\tau_\infty(\lambda))\}_\infty, \end{aligned} \tag{4.29}$$

where $'$ *denotes differentiation with respect to* z*, and*

$$\tau_\infty(\lambda) := \begin{cases} \max^l_{\substack{\nu,\mu=0 \\ v\neq\mu}} (1 + |Re(\lambda)|)^{-1} & \text{if } l > 0, \\ |\lambda|^{-1} & \text{if } l = 0. \end{cases}$$

PROOF: The proof can be found in [MM03, §2.8] or [MM86, Chapter 3]. The existence of $\Psi_i(z) \in M_n(W^1_\infty(a,b))$ follows, since $r_\nu \neq r_\mu$, from

$$\begin{aligned} &\Psi_0(z)A_1 = A_1\Psi_0(z) \\ \Rightarrow &\begin{pmatrix} r_0\Psi_{0,00} & r_1\Psi_{0,01} & \dots & r_l\Psi_{0,0l} \\ r_0\Psi_{0,10} & r_1\Psi_{0,11} & & \vdots \\ \vdots & & \ddots & \\ r_0\Psi_{0,l0} & r_1\Psi_{0,l1} & \dots & r_l\Psi_{0,ll} \end{pmatrix} = \begin{pmatrix} r_0\Psi_{0,00} & r_0\Psi_{0,01} & \dots & r_0\Psi_{0,0l} \\ r_1\Psi_{0,10} & r_1\Psi_{0,11} & & \vdots \\ \vdots & & \ddots & \\ r_l\Psi_{0,l0} & r_l\Psi_{0,l1} & \dots & r_l\Psi_{0,ll} \end{pmatrix} \\ \Rightarrow &\Psi_0(z) = \begin{pmatrix} \Psi_{0,00} & 0 & \dots & 0 \\ 0 & \Psi_{0,11} & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \Psi_{0,ll} \end{pmatrix}, \ \Psi_0(a) = \begin{pmatrix} I_{n_0} & 0 & \dots & 0 \\ 0 & I_{n_1} & & \vdots \\ \vdots & \ddots & & \\ 0 & \dots & & I_{n_l} \end{pmatrix}. \end{aligned}$$

By letting $\Psi_{0,\nu\nu} \in M_n\left(W^1_\infty(a,b)\right)$ be a fundamental matrix of $y' - A_{0,\nu\nu}y = 0$ with $\Psi_{0,\nu\nu}(a) = I_{n_\nu}$, i.e., $\Psi'_{0,\nu\nu} = A_{0,\nu\nu}\Psi_{0,\nu\nu}$ and, for $\nu \neq \mu$, setting $\Psi_{0,\nu\mu} = 0$ we obtain that Ψ_0 satisfies (4.28). ■

Remark 4.5. From the estimates (4.29) and since $\Psi_0(a) = I$ it is easy to see that the above fundamental matrix $\widehat{\Psi}(z,\lambda)$ approximates its corresponding transition matrix for λ large enough. ♣

Corollary 4.6: *Let the conditions of Theorem 4.4 hold. If the block-diagonal elements of A_0 are zero, i.e., $A_{0,\nu\nu} = 0$ $v = 0, 1, \ldots, l$, then we have that $\Psi_0(z) = I$ and*

$$\lim_{\lambda\to\infty} \widehat{\Psi}(z,\lambda) = \lim_{\lambda\to\infty} \left(I + \widehat{\Theta}(z,\lambda)\right) E(z,\lambda) = E(z,\lambda).$$

Thus, under these conditions, the fundamental solution of the system $\frac{d\phi}{dz} = \left(\lambda A_1 + A_0\right)\phi$ converges to the fundamental solution of the system $\frac{d\phi}{dz} = \lambda A_1 \phi$ when $|\lambda| \to \infty$. ♠

PROOF: That $\Psi_0(z) = I$ follows easily since $\Psi_0(z)$ is a block-diagonal matrix where the block diagonal elements are the fundamental solution of $\Psi'_{0,\nu\nu} = A_{0,\nu\nu}\Psi_{0,\nu\nu}$. The other part of the proof follows from $\widehat{\Theta}(z,\lambda) = \{\mathrm{o}(1)\}_\infty$ and $\frac{1}{\lambda}\widehat{\Theta}'(z,\lambda) = \{\mathrm{o}(1)\}_\infty$. ■

Remark 4.7. Note that the condition on the corollary above is on the block diagonal elements of A_0 and not on P_0. ♣

Corollary 4.6 could be useful when trying to prove the exponential stability of some systems. By looking at the expression for A_0, see (4.21), one may wonder whether it is possible to select R in such a way that the diagonal elements are zero. The following theorem gives an answer to this question.

Theorem 4.8: *Consider the problem* (4.21)–(4.22)*. Let R_0 contain the eigenvectors of $(\mathcal{L}P_1)^{-1}$ and Υ be a diagonal matrix whose diagonal elements, denoted by υ_k, satisfy*

$$\upsilon_k\,\vartheta_k - \upsilon'_k = 0 \quad k = 1, \ldots, n,$$

where a prime denotes differentiation with respect to z and ϑ_k is the k-th diagonal element of $-R_0^{-1}(P_1^{-1}P_0R_0 + R'_0)$. Then $R = R_0\,\Upsilon$ satisfy (4.21) *and, moreover, the diagonal elements of A_0 are zero.*

PROOF: It is clear that $R_0^{-1}(\mathcal{L}P_1)^{-1}R_0 = A_1$ with A_1 a diagonal matrix containing the eigenvalues of $(\mathcal{L}P_1)^{-1}$ since R_0 contains the eigenvectors of $(\mathcal{L}P_1)^{-1}$. It is also easy to see that Υ is invertible, i.e., $\upsilon_k \neq 0$. Then, since Υ and A_1 are diagonal, it clearly follows that

$$\Upsilon^{-1}R_0^{-1}(\mathcal{L}P_1)^{-1}\underbrace{R_0\Upsilon}_{R} = \Upsilon^{-1}\,A_1\,\Upsilon = A_1.$$

Next we study the expression for A_0 in (4.21) with $R = R_0\,\Upsilon$. That is

$$\begin{aligned} A_0 &= -\Upsilon^{-1}R_0^{-1}\big(P_1^{-1}P_0R_0\Upsilon + (R_0\Upsilon)'\big) \\ &= -\Upsilon^{-1}R_0^{-1}\big(P_1^{-1}P_0R_0\Upsilon + R'_0\,\Upsilon + R_0\,\Upsilon'\big) \\ &= \Upsilon^{-1}\Big(-R_0^{-1}\big(P_1^{-1}P_0R_0 + R'_0\big)\Big)\Upsilon - \Upsilon^{-1}\,\Upsilon'. \end{aligned}$$

From this it is easy to see that the diagonal elements of A_0 satisfy

$$[A_0]_{kk} = \vartheta_k - \frac{\upsilon_k'}{\upsilon_k}, \quad k = 1, \dots, n.$$

Clearly, by selecting $\upsilon_k = \exp\left(\int_0^z \vartheta_k(\xi)\, d\xi\right)$ we can make the diagonal elements of A_0 equal to zero. ■

4.4. Eigenvalues of $\mathcal{A}_{\mathcal{L}}$

Following the previous section, it is easy to prove the following corollary.

Corollary 4.9: *Let* $\widehat{\Psi}(z,\lambda)$ *be a fundamental matrix of equations* (4.21)–(4.22). *Then* $M(z,\lambda) = R(z)\widehat{\Psi}(z,\lambda)$ *is a fundamental matrix of* (4.11)*, where* $R(z)$ *is given in* (4.21)*. In addition, if* $\widehat{\Psi}(z,\lambda)$ *is a transition matrix of* (4.21)–(4.22)*, then*

$$M(z,\lambda) = R(z)\widehat{\Psi}(z,\lambda)R^{-1}(z) \tag{4.30}$$

is a transition matrix of (4.11). ♠

Following Theorem 4.2 we have that the eigenvalues of $\mathcal{A}_{\mathcal{L}}$ are given by the zeros of the characteristic determinant

$$\Delta(\lambda) = \det H_0(\lambda). \tag{4.31}$$

Using Theorem 4.4 we obtain an asymptotic representation for this characteristic determinant, see also Proposition 2.6 of [Tre00b] or [MM86, §5]. Here a function of the form

$$[f(\lambda)]_k = \sum_{j=0}^{k} \lambda^{-j} f_j + \lambda^{-k} \mathrm{o}(1)$$

is called an *asymptotic polynomial of order k with respect to λ^{-1}*.

Theorem 4.10: *Let* $M(z,\lambda)$ *be a fundamental matrix of* (4.11) *as in Corollary* 4.9 *and Theorem* 4.4*. Then, for sufficiently large* $\lambda \in \mathbb{C}$*, i.e.,* $|\lambda| > K$*, the characteristic determinant* Δ *of* (4.11) *and* (4.13) *has the asymptotic expansion*

$$\Delta(\lambda) = \sum_{c\in\mathcal{E}} \left(b_c + \{o(1)\}_\infty\right) \mathrm{e}^{\lambda c}. \tag{4.32}$$

Here $b_c \in \mathbb{R}$ *and* $\mathcal{E}$ *is the set*

$$\mathcal{E} = \left\{ \sum_{\nu=1}^{n} \delta_\nu R_\nu(b) \;\middle|\; \delta_\nu = \{0,\, 1\} \right\} \subset \mathbb{R},$$

where R_ν, $\nu = 1, 2, \ldots, n$, are given by (4.25). Furthermore, the nonzero coefficients b_c correspond to the determinant of $W_b R(b)\Psi_0(b)E(b,\lambda) + W_a R(b)$, i.e.,

$$\sum_{c\in\mathcal{E}} b_c\, \mathrm{e}^{\lambda c} = \det\bigl(W_b R(b)\Psi_0(b)E(b,\lambda) + W_a R(b)\bigr). \tag{4.33}$$

PROOF: Basically, we only need to replace $\widehat{\Psi}(z,\lambda)$ in Theorem 4.4 by the expression for $M(z,\lambda)$ given in Corollary 4.9, and use this in $H_0(\lambda)$, see Theorem 4.2. That is,

$$\begin{aligned} H_0(\lambda) &= W_b\, M(b,\lambda) + W_a \\ &= W_b\, R(b)\left(\Psi_0(b) + \widehat{\Theta}(b,\lambda)\right) E(b,\lambda) R^{-1}(b) + W_a. \end{aligned}$$

Since $\widehat{\Theta}(z,\lambda) = \{o(1)\}_\infty$ and $E(z,\lambda)$ is diagonal, we can see that each element of $H_0(\lambda)R(b)$ can be written as $[s^b_{\nu\mu}]_0\, \mathrm{e}^{\lambda R_\mu(b)} + s^a_{\nu\mu}$ where $[s^b_{\nu\mu}]_0 = s^b_{\nu\mu} + \{o(1)\}_\infty$, $s^b_{\nu\mu} \in \mathbb{R}$, is an asymptotic polynomial of order 0. Here, $s^b_{\nu\mu}$ is the $\nu\mu$-th component of the matrix $W_b R(b)\Psi_0(b)$ and $s^a_{\nu\mu}$ is the $\nu\mu$-th element of $W_a R(b)$. Recall that $R_\mu(b) = \int_a^b r_\mu(\xi)\, d\xi$, see (4.25). Thus,

$$\det H_0(\lambda) = k \det\left([s^b_{\nu\mu}]_0\, \mathrm{e}^{\lambda R_\mu(b)} + s^a_{\nu\mu}\right)^n_{\nu,\mu=1},$$

where $k = 1/\det(R(b))$ is a constant. Next we use the definition of the determinant, see [Mey01]. To do so, let $p = (p1, p2, \ldots, pn)$ be a permutation of $(1, 2, \ldots, n)$ with $\sigma(p)$ being the corresponding sign of this permutation. Also, let $e_{pi,z} = \mathrm{e}^{\lambda R_{pi}(z)}$ and $e_{p1_{z_1},p2_{z_2},\ldots,pk_{z_k}} = \mathrm{e}^{\lambda[R_{p1}(z_1)+R_{p2}(z_2)+\cdots+R_{pk}(z_k)]}$. Thus,

$$\begin{aligned} \det(H_0(\lambda)) &= k \det\Bigl([W_b R(b)\Psi_0(b) + o(1)]E(b,\lambda) + W_a R(b)\Bigr) \\ &= k\sum_p \sigma(p)[(s^b_{1,p1} + o(1))e_{p1,b} + s^a_{1,p1}e_{p1,a}]\cdots[(s^b_{n,pn} + o(1))e_{pn,b} + s^a_{n,pn}\, e_{pn,a}] \\ &= k\sum_p \sigma(p)\Big[(s^b_{1,p1}s^b_{2,p2} + o(1))e_{p1_b,p2_b} + (s^b_{1,p1}s^a_{2,p2} + o(1))e_{p1_b,p2_a} \\ &\qquad + (s^a_{1,p1}s^b_{2,p2} + o(1))e_{p1_a,p2_b} + (s^a_{1,p1}s^a_{2,p2})e_{p1_a,p2_a}\Big]\cdots \\ &\qquad \cdots\Big[(s^b_{n,pn} + o(1))e_{pn,b} + (s^a_{n,pn} + o(1))e_{pn,a}\Big] \\ &= k\sum_p \sigma(p)\Big[\Big(s^b_{1,p1}s^b_{2,p2}\cdots s^b_{n,pn} + o(1)\Big)e_{p1_b,\ldots,pn_b} \\ &\qquad + \Big(s^b_{1,p1}\cdots s^b_{n-1,p(n-1)}s^a_{n,pn} + o(1)\Big)e_{p1_b,\ldots,p(n-1)_b,pn_a} \end{aligned}$$

$$+ \cdots + \left(s^a_{1,p1} \cdots s^a_{n-1,p(n-1)} s^b_{n,pn} + o(1)\right) e_{p1_a,\ldots,p(n-1)_a,pn_b}$$
$$+ \left(s^a_{1,p1} \cdots s^a_{n-1,p(n-1)} s^a_{n,pn}\right) e_{p1_a,\ldots,p(n-1)_a,pn_a}\Big],$$

where we used $\kappa o(1) = o(1)$ for any constant κ. Recall that $R_{pi}(a) = 0$ and hence $e_{pi,a} = 1$. From all this the result follows. ■

The next definitions are well-known, we recall it for the sake of completeness.

Definition 4.11. An entire function $F(\cdot)$ is said to be of *exponential type* if the inequality

$$|F(z)| \le C\mathrm{e}^{L|z|}$$

holds for some positive constants C and L and all complex values of z.

A point $z_0 \in \mathbb{C}$ such that $F(z_0) = 0$ is called a *zero* of the entire function F. The integer l such that $F(z_0) = F'(z_0) = \cdots = F^{(l)}(z_0) = 0$ but $F^{(l+1)}(z_0) \neq 0$ is called the *vanishing order* of F. We say z_0 is a *simple zero* of F if $l = 0$, otherwise, it is called a *multiple zero*. An entire function of exponential type F is said to be of *sine-type* if

(a) the zeros of F lie in a strip $\{z \in \mathbb{C} \mid |\mathrm{Re}\, z| \le c\}$ for some $c > 0$;

(b) there exist constants $c_1,\, c_2 > 0$ and $x_0 \in \mathbb{R}$ such that $c_1 \le |F(x_0 + iy)| \le c_2$ for all $y \in \mathbb{R}$. ♣

The following definition appears (implicitly or explicitly) often as an assumption in applications.

Definition 4.12. Let $\mathcal{E}$ be the set obtained in Theorem 4.10 and define w_M and w_m by

$$w_M = \max_{c \in \mathcal{E}} \mathcal{E}, \qquad w_m = \min_{c \in \mathcal{E}} \mathcal{E},$$

and let $[b_M]_0 = b_{M0} + \{o(1)\}_\infty$ and $[b_m]_0 = b_{m0} + \{o(1)\}_\infty$ be the corresponding coefficients in (4.32), respectively. The problem (4.11) and (4.13) is called *normal* if $b_{M0} \neq 0$ and $b_{m0} \neq 0$. ♣

The importance of the definition above is that it guarantees that the spectrum is bounded in some directions.

From Lemma 5.12 of [MM86] we obtain the following result.

Lemma 4.13: *Let the problem* (4.11) *and* (4.13) *be normal. Then, there is a positive constant* δ *and a* $w \in \mathcal{E}$ *(*w *may depend on* λ*) such that Re*$\,(\lambda(c-w)) \le 0$ *for* $c \in \mathcal{E}$ *and*

$$\lim_{Re\,\lambda\to\pm\infty} |\mathrm{e}^{-w\lambda}\Delta(\lambda)| \ge \delta.$$

In addition, $\Delta(\lambda)$ *is a sine-type function. As a consequence, the vanishing orders of* Δ *at its zeros are uniformly bounded and all its zeros lie in a vertical strip parallel to the imaginary axis.* ♡

PROOF: Notice that $\Delta(\lambda)$ is of exponential type and that the elements of $\mathcal{E}$ are real, i.e., $c \in \mathbb{R}$. Following the notation introduced in Definition 4.12 we get

$$\begin{aligned}\lim_{\mathrm{Re}\,\lambda\to\infty} \left|\mathrm{e}^{-\lambda w_M}\Delta(\lambda)\right| &= \lim_{\mathrm{Re}\,\lambda\to\infty} |[b_M]_0| = |b_{M0}| > 0,\\ \lim_{\mathrm{Re}\,\lambda\to-\infty} \left|\mathrm{e}^{-\lambda w_m}\Delta(\lambda)\right| &= \lim_{\mathrm{Re}\,\lambda\to-\infty} |[b_m]_0| = |b_{m0}| > 0.\end{aligned} \tag{4.34}$$

Therefore, there exists a constant $M > 0$ such that $\Delta(\lambda) \ne 0$ as $|\mathrm{Re}\,\lambda| > M$. Moreover, it is easy to see that $|\Delta(x_0+iy)|$ is bounded below and above for some $|x_0| > M$, see (4.34). Hence it is clear by definition that $\Delta(\lambda)$ is a sine-type function. That the vanishing orders of Δ at its zeros are uniformly bounded follows from [AI95, Proposition II.1.28] or [GX06, Proposition 2.1]. This completes the proof. ■

Example 4.14 Consider the wave equation, which can be modeled by (see Example 1.6)

$$\frac{\partial}{\partial t}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial z}\begin{bmatrix} \frac{1}{\rho}p \\ Tq \end{bmatrix}, \quad t \ge 0, \tag{4.35}$$

$$T(a)q(a,t) = 0, \ \frac{1}{\rho(b)}p(b,t) + T(b)q(b,t) = 0, \tag{4.36}$$

where $z \in [a,b]$ is the spatial variable, $q(z,t) = \frac{\partial u}{\partial z}(z,t)$ is the strain, $p(z,t) = \rho\frac{\partial u}{\partial t}(z,t)$ is the momentum distribution, $u(z,t)$ is the displacement, $T(z)$ is the Young's modulus, and $\rho(z)$ is the mass density. Here $0 < T_m < T(z) < T_M$ and $0 < \rho_m < \rho(z) < \rho_M$ with T_m, T_M, ρ_m, ρ_M (positive) constants. Let $e = \left[\begin{smallmatrix} e_p \\ e_q \end{smallmatrix}\right] = \left[\begin{smallmatrix} \frac{1}{\rho}p \\ Tq \end{smallmatrix}\right]$, i.e., e_p and e_q are the velocity and the stress, respectively. Then the port-variables are

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}\begin{bmatrix} e(b) \\ e(a) \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} e_q(b)-e_q(a) \\ e_p(b)-e_p(a) \\ e_p(b)+e_p(a) \\ e_q(b)+e_q(a) \end{bmatrix} = \begin{bmatrix} f_{\partial_1} \\ f_{\partial_2} \\ e_{\partial_1} \\ e_{\partial_2} \end{bmatrix}. \tag{4.37}$$

To the boundary conditions (4.36) there correspond a $W = \begin{bmatrix} W_1, & W_2 \end{bmatrix}$ matrix given by

$$W_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad W_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

From this and equation (4.10c) we obtain

$$W_b = \sqrt{2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad W_a = \sqrt{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We want to find the eigenvalues of this system, but since the parameters ρ and T depend on z it is very hard to find an exact expression for them. We will use the approach presented here to find an asymptotic approximation of the eigenvalues of the system. From Theorem 4.10 we see that we need first to find the matrices R and Ψ_0, see (4.33). We also use Theorem 4.8 in order to make the diagonal elements of A_0 zero. Since the diagonal elements of $-R_0^{-1}R_0'$ (note $P_0 = 0$) are $\frac{1}{4}(T\rho)'/(T\rho)$ (a prime $'$ denotes differentiation with respect to z), in this case, $R_0(z)$ and $\Upsilon(z)$ are given by

$$R_0(z) = \begin{bmatrix} 1 & 1 \\ \sqrt{(T\rho)(z)} & -\sqrt{(T\rho)(z)} \end{bmatrix}, \; \Upsilon(z) = \exp\left(-\frac{1}{4}\ln\big((T\rho)(z)\big)\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $R = R_0\,\Upsilon$ is

$$R = \exp\left(-\frac{1}{4}\ln\big((T\rho)(z)\big)\right)\begin{bmatrix} 1 & 1 \\ \sqrt{(T\rho)(z)} & -\sqrt{(T\rho)(z)} \end{bmatrix}.$$

This selection of R guarantees that the diagonal elements (only the diagonal elements) are zero, see Theorem 4.8. This is all we need to calculate about A_0 since only its diagonal elements are needed in the subsequent calculations. From this selection of R and (4.21) we get (for simplicity we drop the dependance on the variables z and λ)

$$A_1 = \sqrt{\frac{\rho}{T}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow E = \begin{bmatrix} \exp\left(\lambda \int_a^z \sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi\right) & 0 \\ 0 & \exp\left(-\lambda \int_a^z \sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi\right) \end{bmatrix}$$

and $[A_0]_{\nu\nu} = 0$, $\nu = 1, 2$. It thus follows that in Theorem 4.4 $\Psi_0 = I$. Note that the set $\mathcal{E}$ appearing in Theorem 4.10, in this case, is given by (noting that $R_1(b) = \int_a^b \sqrt{\rho(\xi)/T(\xi)}\,d\xi$ and $R_2(b) = -\int_a^b \sqrt{\rho(\xi)/T(\xi)}\,d\xi$, see (4.25))

$$\begin{aligned} \mathcal{E} &= \{R_1(b) + R_2(b),\, R_1(b),\, R_2(b),\, 0\} = \{0,\, R_1(b),\, R_2(b),\, 0\} \\ &= \left\{\int_a^b \sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi,\; -\int_a^b \sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi,\; 0\right\}. \end{aligned} \tag{4.38}$$

From (4.33) we get

$$\begin{aligned} &\det\Big(W_b R(b)\Psi_0(b)E(b,\lambda) + W_a R(b)\Big) \\ &\quad = k_b\Big(1 + \sqrt{(T\,\rho)(b)}\Big)E_1(b,\lambda) + k_b\Big(1 - \sqrt{(T\,\rho)(b)}\Big)E_2(b,\lambda) \end{aligned} \tag{4.39}$$

where $E_i(b,\lambda)$, $i=1,2$, is the i-th diagonal element of $E(b,\lambda)$ and $k_b = 2\sqrt{T\,\rho}(b)$. Following Definition 4.12 and (4.38) we can see that $w_M = \int_a^b \sqrt{\rho(\xi)/T(\xi)}\,d\xi$ and $w_m = -\int_a^b \sqrt{\rho(\xi)/T(\xi)}\,d\xi$. Thus, the problem will be normal if the respective coefficients of $\mathrm{e}^{\lambda w_M}$ and $\mathrm{e}^{\lambda w_m}$ in Δ are nonzero. From the equation above, we can see it is equivalent to $k_b\Big(1+\sqrt{(T\,\rho)(b)}\Big)$ and $k_b\Big(1-\sqrt{(T\,\rho)(b)}\Big)$ being different than zero, i.e., $1 \neq \sqrt{(T\,\rho)(b)}$. In fact, again from Theorem 4.10, we get that the characteristic determinant has the asymptotic expansion (for λ large enough)

$$\begin{aligned}\Delta(\lambda) = \Big(k_b\big(1+\sqrt{(T\,\rho)(b)}\big) + \{o(1)\}_\infty\Big)\exp\left(\lambda\int_a^b\sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi\right) + \{o(1)\}_\infty\exp(0)\\ + \Big(k_b\big(1-\sqrt{(T\,\rho)(b)}\big) + \{o(1)\}_\infty\Big)\exp\left(-\lambda\int_a^b\sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi\right).\end{aligned} \tag{4.40}$$

We know that the roots of Δ are the eigenvalues of the system, see Theorem 4.2. By Rouché's theorem, the roots of Δ can be estimated by those of (4.39), that is, by the roots of $\det\Big(W_b R(b)\Psi_0(b)E(b,\lambda) + W_a R(b)\Big)$. These roots can be found explicitly, and are given by

$$\tilde{\lambda}_m = \begin{cases} -\frac{1}{2\int_a^b\sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi}\ln\left|\frac{\sqrt{(T\,\rho)(b)}+1}{\sqrt{(T\,\rho)(b)}-1}\right| - i\frac{\pi\,m}{\int_a^b\sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi} & \text{if } \sqrt{(T\,\rho)(b)} > 1 \\ -\frac{1}{2\int_a^b\sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi}\ln\left|\frac{\sqrt{(T\,\rho)(b)}+1}{\sqrt{(T\,\rho)(b)}-1}\right| - i\frac{\pi(1+2m)}{2\int_a^b\sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi} & \text{if } \sqrt{(T\,\rho)(b)} < 1 \end{cases} \quad m\in\mathbb{Z}$$

Thus, the roots of (4.40) satisfy (by Rouché's theorem)

$$\lambda_m = \tilde{\lambda}_m + O(m^{-1}), \quad |m| > N_1,\ m\in\mathbb{Z}, \tag{4.41}$$

where N_1 is some sufficiently large positive integer. $\ast$

4.5. Minimality, completeness, and Riesz basis property

In this section we investigate minimality and completeness properties of the eigenfunctions of the problem (4.11)–(4.13). We do it by proving all this for the auxiliary eigenvalue problem (4.21)–(4.23). Related to the eigenvalue problem (4.21)–(4.22) we can define the operator $\mathcal{A}_\phi$ as follows

$$\mathcal{A}_\phi y = A_1^{-1}\frac{\partial y}{\partial z} - A_1^{-1}A_0 y = R^{-1}\mathcal{L}P_1\frac{\partial Ry}{\partial z} + R^{-1}\mathcal{L}P_0(Ry), \tag{4.42a}$$

with domain

$$D(\mathcal{A}_\phi) = \left\{ Ry \in H^1(a,b;\mathbb{R}^n) \mid W_b(Ry)(b) + W_a(Ry)(a) = 0 \right\}. \tag{4.42b}$$

It is easy to see that the eigenvalue problem related to the operator $\mathcal{A}_\phi$ is the same as the eigenvalue problem (4.21)–(4.22). That is, if λ is an eigenvalue of $\mathcal{A}_\phi$ with corresponding eigenvector ϕ then these λ and ϕ satisfy (4.21)–(4.22), and viceversa. From this, we can easily obtain the eigenvalues and eigenvectors of $\mathcal{A}_\mathcal{L}$ with the help of the following lemma.

Lemma 4.15: *Consider the operator* $\mathcal{A}_\phi$ *defined by* (4.42) *and the operator* $\mathcal{A}_\mathcal{L}$ *defined by* (4.10). *Then,*

i) $y \in D(\mathcal{A}_\phi)$ *if and only if* $x = \mathcal{L}^{-1}Ry \in D(\mathcal{A}_\mathcal{L})$, *or equivalently,* $y = R^{-1}\mathcal{L}x \in D(\mathcal{A}_\phi)$ *if and only if* $x \in D(\mathcal{A}_\mathcal{L})$,

ii) If λ *is an eigenvalue of* $\mathcal{A}_\phi$ *with corresponding eigenvector* y. *Then* λ *is an eigenvalue of* $\mathcal{A}_\mathcal{L}$ *with corresponding eigenvector* $x = \mathcal{L}^{-1}Ry$.

iii) If λ *is an eigenvalue of* $\mathcal{A}_\mathcal{L}$ *with corresponding eigenvector* x. *Then* λ *is an eigenvalue of* $\mathcal{A}_\phi$ *with corresponding eigenvector* $y = R^{-1}\mathcal{L}x$. ♡

PROOF: *i*) Let $y \in D(\mathcal{A}_\phi)$. It thus satisfies $W_b(Ry)(b) + W_a(Ry)(a) = 0$. By letting $Ry = \mathcal{L}x$, it is clear that $x = \mathcal{L}^{-1}Ry \in D(\mathcal{A}_\mathcal{L})$. The other part of the proof follows the same idea.

ii) Let λ be an eigenvalue of $\mathcal{A}_\phi$ with corresponding eigenvector y. It thus follows, by using (4.21), that

$$\begin{aligned}
\lambda y &= A_1^{-1}\frac{\partial y}{\partial z} - A_1^{-1}A_0 y = R^{-1}\mathcal{L}P_1R\frac{\partial y}{\partial z} + (R^{-1}\mathcal{L}P_1R)(R^{-1}P_1^{-1}P_0R + R^{-1}R')y \\
&= R^{-1}\mathcal{L}P_1R\frac{\partial y}{\partial z} + R^{-1}\mathcal{L}P_0Ry + R^{-1}\mathcal{L}P_1R'y \\
&= R^{-1}\mathcal{L}P_1\frac{\partial Ry}{\partial z} + R^{-1}\mathcal{L}P_0(Ry) \qquad \text{(this justifies (4.42a)).}
\end{aligned}$$

By letting $\mathcal{L}x = Ry$ yields $\lambda x = \mathcal{A}_\mathcal{L}x$.

iii) Let λ be an eigenvalue of $\mathcal{A}_\mathcal{L}$ with corresponding eigenvector x. It thus follows, by letting $\mathcal{L}x = Ry$, that

$$\begin{aligned}
\lambda x &= P_1\frac{\partial \mathcal{L}x}{\partial z} + P_0(\mathcal{L}x) \\
\Rightarrow \lambda R^{-1}\mathcal{L}x &= R^{-1}\mathcal{L}P_1\frac{\partial \mathcal{L}x}{\partial z} + R^{-1}\mathcal{L}P_0(\mathcal{L}x) \\
\Rightarrow \lambda y &= R^{-1}\mathcal{L}P_1\frac{\partial Ry}{\partial z} + R^{-1}\mathcal{L}P_0(Ry) = \mathcal{A}_\phi y.
\end{aligned}$$

From this the result follows. ■

Following this we can study the minimality and completeness properties of the eigenfunctions of the problem (4.21)–(4.23) and from there we can obtain the same results for the operator $\mathcal{A}_{\mathcal{L}}$. To do so, we introduce the related holomorphic operator function T on $\mathbb{C}$ given by

$$T(\lambda) = T_0 - \lambda T_1, \tag{4.43}$$

where the bounded linear operators $T_0,\, T_1 \in \mathcal{L}(H^1(a,b)^n,\, L_2(a,b)^n \times \mathbb{C}^n)$ are

$$T_0 y := \begin{bmatrix} y' - A_0 y \\ W_b(Ry)(b) + W_a(Ry)(a) \end{bmatrix}, \quad T_1 y := \begin{bmatrix} A_1 y \\ 0 \end{bmatrix}, \tag{4.44}$$

with W_b and W_a defined by (4.10c). Note that (4.43)–(4.44) corresponds to the eigenvalue problem related to $\mathcal{A}_\phi$. For the operator T we define the resolvent set of T as

$$\rho(T) := \{\lambda \in \mathbb{C} : T(\lambda) \text{ is invertible }\},$$

$\sigma(T) = \mathbb{C}\backslash\rho(T)$ is the spectrum of T, and T^{-1} is called the resolvent of T, see [MM03, §1.2]. Note that, in this case, $T^{-1}T = I_{H^1}$ and $TT^{-1} = I_{L_2\times\mathbb{C}}$, where I_Y is the identity operator on the space Y. λ_ν is said to be an eigenvalue of (4.21)–(4.23), i.e., of $\mathcal{A}_\phi$, if λ_ν is in the point spectrum of T, i.e., $\lambda_\nu \in \sigma_p(T)$, and $\{y_\nu^s\}_{s=0}^{p_\nu-1} \subset H^1(a,b)^n$ is called a chain of eigenfunction and associated functions of (4.21)–(4.23) at λ_ν if it is a chain of an eigenfunction and associated functions of T at λ_ν, i.e., y_ν^0 is an eigenvector of T at λ_ν and for

$$y_\nu(\lambda) = \sum_{s=0}^{p_\nu-1} (\lambda-\lambda_\nu)^s y_\nu^s$$

the function Ty_ν has a zero of order greater than or equal to p_ν at λ_ν. If μ is an eigenvalue of finite algebraic multiplicity, then a system $\{y_j^s\}_{s=0,\, j=1}^{p_j-1,\, r}$ is called a canonical system of eigenfunctions and associated functions of (4.21)–(4.23) at μ if it is a canonical system of eigenfunctions and associated functions of T at μ, i.e.,

i) $\{y_1^0, \dots, y_r^0\}$ is a basis of $\ker T(\mu)$,

ii) $\{y_j^s\}_{s=0}^{p_j-1}$ is a maximal chain of an eigenfunction and associated functions for $j = 1, 2, \dots, r$,

iii) $p_j = \sup\{p(\mu, x_0) \mid x_0 \in \ker T(\mu)\backslash \operatorname{span}\{y_k^0 \mid k < j\}\}$, $j = 1, \dots, r$, where $p(\mu, x_0)$ denotes the rank of an eigenvector x_0.

In this case, a chain $\{y_\nu^s\}_{s=0}^{p_\nu-1}$ of an eigenfunction and associated functions of the eigenvalue problem (4.23), and hence of T, at λ_ν fulfills the relations

$$T_0 y_\nu^s - \lambda_\nu T_1 y_\nu^s = T_1 y_\nu^{s-1}, \quad s = 0, 1, \dots, p_\nu - 1, \quad y_\nu^{-1} := 0.$$

It is well-known that $T(\lambda)$ is Fredholm with index zero for each $\lambda \in \mathbb{C}$, [MM86]. Therefore, if the resolvent set $\rho(T)$ is non-empty, $\sigma(T)$ is a discrete subset of $\mathbb{C}$,

$\sigma(T) = \sigma_p(T)$, all eigenvalues are of finite algebraic multiplicity and accumulate only at infinity (see [MM86, Chapter 7]). Notice that if λ_ν is an eigenvalue of T, and hence of $\mathcal{A}_\phi$, then it is an eigenvalue of $\mathcal{A}_\mathcal{L}$ by Lemma 4.15; and the eigenvectors are related by the same lemma.

ASSUMPTION 4.16: Throughout this section we will assume that the eigenvalue problem (4.11) and (4.13) is normal, see Definition 4.12. ♡

Here we want to use the results of [Tre00b] and [Tre00a]. There, the author assumes that the problem (4.21)–(4.23) is non-degenerate, i.e., $\rho(T) \neq \emptyset$. Observe that here we have that $\mathbb{C}_+ \subset \rho(T)$, cf. Theorem 4.2, and thus that condition is satisfied. Without loss of generality we assume that $0 \in \rho(T)$, that is, T_0 is bijective. Otherwise we substitute T_0 by $T_0 - \lambda_0 T_1$ and λ by $\lambda - \lambda_0$, with $\lambda_0 \in \rho(T)$. Observe that this substitution changes A_0 to $A_0 + \lambda_0 A_1$ and since we do not make any assumptions on A_0 it is obvious that the assumption $0 \in \rho(T)$ is valid.

Remark 4.17. It is worth to mention that $0 \in \rho(T) \iff 0 \in \rho(\mathcal{A}_I)$, with $\mathcal{L} = I$. ♣

From Lemma 1.1 of [Tre00a] we can see that the linear pencil $K_r := I - \lambda T_1 T_0^{-1}$ is equivalent to $T = T_0 - \lambda T_1$, i.e., $\rho(T) = \rho(K_r)$, $\sigma(T) = \sigma(K_r)$, $\sigma_p(T) = \sigma_p(K_r)$. Also, it is easy to prove that T_1 is a compact operator, see [Tre00b, Lemma 3.1].

Definition 4.18. Let G be a Banach space. A system $\{e_i\}_1^\infty \subset G$ is called *minimal* if there exists a system $\{f_j\}_1^\infty \subset G'$ (the dual of G) which is biorthogonal to $\{e_i\}_1^\infty$:

$$\langle e_i, f_j \rangle := f_j(e_i) = \delta_{ij}, \quad i, j = 1, 2, \ldots .$$

The system $\{e_i\}_1^\infty$ is called *minimal with defect* m, $m \in \mathbb{N}_0$, if there exists m elements $e_{i_1}, \ldots, e_{i_m} \subset \{e_i\}_1^\infty$ such that $\{e_i\}_1^\infty \backslash \{e_{i_1}, \ldots, e_{i_m}\}$ is minimal and

$$\overline{\operatorname{span}\left(\{e_i\}_1^\infty \backslash \{e_{i_1}, \ldots, e_{i_m}\}\right)} = \overline{\operatorname{span}\{e_i\}_1^\infty}.$$

The number m is called the defect of minimality. ♣

The following theorem is an adaptation of Theorem 2.4 of [Tre00a], where we use the fact that T_1 is compact.

Theorem 4.19: *Consider the operators* $T_0, T_1 \in \mathcal{L}(H^1(a,b)^n, L_2(a,b)^n \times \mathbb{C}^n)$ *and let* $\{y_\nu^s\}$ *be a canonical system of eigenvectors and associated vectors of the linear pencil* $T(\lambda) = T_0 - \lambda T_1$*. Then*

i) $\{y_\nu^s\}$ *is minimal in* $H^1(a,b)^n$*,*

ii) $\{T_1 y_\nu^s\}$ *is minimal in* $L_2(a,b)^n \times \mathbb{C}^n$*,* *and there exists a canonical system* $\{v_\mu^t\} \subset L_2(a,b)^n \times \mathbb{C}^n$ *of eigenvectors and associated vectors of the adjoint linear pencil* $T^*(\lambda) = T_0^* - \lambda T_1^*$ *such that*

iii) $\{-T_1^* v_\mu^{p_\mu - 1 - t}\} \subset H^1(a,b)^n$ *is biorthogonal to* $\{y_\nu^s\}$*,*

iv) $\{-v_\mu^{p_\mu - 1 - t}\} \subset L_2(a,b)^n \times \mathbb{C}^n$ *is biorthogonal to* $\{T_1 y_\nu^s\}$*,*

where p_μ *denotes the rank of the eigenvector* y_μ^0 *and* T_1^* *is the Hilbert adjoint operator of* T_1*.*

Theorem 4.20: *Let* $\{y_\nu^s\}$ *be a canonical system of eigenvectors and associated vectors of* (4.21)–(4.23) *and hence of the linear pencil* $T(\lambda) = T_0 - \lambda T_1$*. Let* $\{v_\mu^t\}$ *be a canonical system of eigenvectors and associated vectors of* $T^*(\lambda) = T_0^* - \lambda T_1^*$*, and define* $\mathcal{V} := \overline{\operatorname{span}\{y_\nu^s\}}$*,* $\mathcal{W} := \overline{\operatorname{span}\{v_\mu^s\}}$*. Then*

$$\mathcal{V} = T_0^{-1}\left(\left[\ker\left((T_1 T_0^{-1})^*\right)^2\right]^\perp\right) \quad \textit{and} \quad \mathcal{W} = T_0^{-*}\left(\left[\ker(T_0^{-1}T_1)^2\right]^\perp\right). \tag{4.45}$$

Furthermore, let $\mathcal{P}$ *denote the orthogonal projection of* $L_2(a,b)^n \times \mathbb{C}^n$ *onto* $L_2(a,b)^n$ *and let* $\mathcal{Q} = I - \mathcal{P}$ *denote the orthogonal projection onto* $\mathbb{C}^n$*. Then, the system* $\{\mathcal{P}T_1 y_\nu^s\}$ *and* $\{y_\nu^s\}$ *are complete and minimal with finite defect* m_0 *in* $\mathcal{P}T_1\mathcal{V} = A_1\mathcal{V} \subset L_2(a,b)^n$*, where*

$$m_0 = \operatorname{codim}_{\mathcal{QW}}(\mathcal{QW} \cap \mathcal{Q}(T_1\mathcal{V})^\perp). \tag{4.46}$$

PROOF: Equation (4.45) follows easily from Theorems 3.3 and 3.4 of [Tre00b] by noticing that, in this case, $\kappa_0 = 0$ and $\kappa_2 = 2$ (using the notation of [Tre00b]).

By Theorem 4.19 we know that $\{T_1 y_\nu^s\}$ is minimal in $L_2(a,b)^n \times \mathbb{C}^n$ with biorthogonal system $\{-v_\mu^{p_\mu - 1 - t}\}$. Clearly $\operatorname{codim}\operatorname{ran}(\mathcal{P}) = n < \infty$, which allows us to use Theorem 2.4 of [Tre00a] to conclude that $\{\mathcal{P}T_1 y_\nu^s\}$ is minimal with defect m_0 given by (4.46). The rest of the proof ensue from $\{\mathcal{P}T_1 y_\nu^s\} = \{A_1 y_\nu^s\}$ and the invertibility of A_1, see Assumption 4.3 andequation (4.21). ∎

In order to look more closely the spaces $\mathcal{V}$ and $\mathcal{W}$ in (4.45) we find the operators $T_1 T_0^{-1}$ and $(T_1 T_0^{-1})^*$. The inverse of T_0 can be found as follows. Let $M_0(z)$ be the transition matrix of $y' - A_0 y = 0$ with corresponding Cauchy matrix $M_0(z,\tau)$, see Section 4.1, and let $\left[\begin{smallmatrix} u \\ d \end{smallmatrix}\right] \in L_2(a,b)^n \times \mathbb{C}^n$ be arbitrary. Then

$$T_0 y = \begin{bmatrix} u \\ d \end{bmatrix} \Rightarrow \begin{bmatrix} y' - A_0 y \\ W_b(Ry)(b) + W_a(Ry)(a) \end{bmatrix} = \begin{bmatrix} u \\ d \end{bmatrix}.$$

Using the top equation, we get

$$y(z) = M_0(z)y(a) + \int_a^z M_0(z,\tau)u(\tau)\,d\tau,$$

and substituting this for $y(z)$ in the second relation yields

$$W_bR(b)\left[M_0(b)y(a) + \int_a^b M_0(b,\tau)u(\tau)\,d\tau\right] + W_aR(a)y(a) = d.$$

Thus

$$y(a) = \underbrace{\Big(W_bR(b)\,M_0(b) + W_aR(a)\Big)}_{H_0}^{-1}\left[-W_bR(b)\int_a^b M_0(b,\tau)u(\tau)\,d\tau + d\right], \tag{4.47}$$

provided H_0 is invertible. Clearly $M_0(z)$ also corresponds to the fundamental matrix of $y' - A_0y - \lambda A_1 y = 0$ when $\lambda = 0$, i.e., $M_0(z) = \widehat{\Psi}(z, \lambda = 0)$. Then, applying the same arguments used to prove Theorem 4.2.i (see the paragraphs before that theorem) we can see that the matrix H_0 above is nonsingular since we assumed that $0 \in \rho(T) = \rho(\mathcal{A}_\phi)$.

Altogether gives the following lemma.

Lemma 4.21: *The operator* $T_0^{-1} \in \mathcal{L}(L_2(a,b)^n \times \mathbb{C}^n,\ H^1(a,b)^n)$ *is given by*

$$\left(T_0^{-1}\begin{bmatrix} u \\ d \end{bmatrix}\right)(z) = M_0(z)H_0^{-1}\left[-W_bR(b)\int_a^b M_0(b,\tau)u(\tau)\,d\tau + d\right] + \int_a^z M_0(z,\tau)u(\tau)\,d\tau, \tag{4.48}$$

for any $\left[\begin{smallmatrix} u \\ d \end{smallmatrix}\right] \in L_2(a,b)^n \times \mathbb{C}^n$*, where* H_0 *is given in* (4.47)*. Thus using* (4.44) *and* (4.48) *we obtain*

$$T_1T_0^{-1}\begin{bmatrix} u \\ d \end{bmatrix}(z) = \begin{bmatrix} A_1(z)M_0(z)H_0^{-1}\left[-W_bR(b)\int_a^b M_0(b,\tau)u(\tau)\,d\tau + d\right] \\ 0 \end{bmatrix} + \begin{bmatrix} A_1(z)\int_a^z M_0(z,\tau)u(\tau)\,d\tau \\ 0 \end{bmatrix}. \tag{4.49}$$

♡

PROOF: It can be checked directly that $T_0\left(T_0^{-1}\left[\begin{smallmatrix} u \\ d \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} u \\ d \end{smallmatrix}\right]$ and by using integration by parts it follows that $T_0^{-1}T_0y = y$ for all $y \in H^1(a,b)$. ∎

Lemma 4.22: *The adjoint of the operator* $T_1T_0^{-1}$ *given by* (4.49) *is*

$$(T_1T_0^{-1})^* \begin{bmatrix} v \\ c \end{bmatrix}(z) = \begin{bmatrix} -M_0^T(b,z)\int_a^b R^T(b)W_b^T H_0^{-T} M_0^T(\tau)(A_1v)(\tau)\,d\tau + \int_z^b M_0^T(\tau,z)(A_1v)(\tau)\,d\tau \\ \int_a^b H_0^{-T}M_0^T(z)(A_1v)(z)\,dz \end{bmatrix} \tag{4.50}$$

♡

PROOF: The adjoint of $T_1T_0^{-1}$ is determined by (recall that $A_1(z)$ is symmetric)

$$\begin{aligned}\left\langle (T_1T_0^{-1})^{-*}\begin{bmatrix} v \\ c \end{bmatrix}, \begin{bmatrix} u \\ d \end{bmatrix}\right\rangle_{L_2\times\mathbb{C}} &= \left\langle \begin{bmatrix} v \\ c \end{bmatrix}, (T_1T_0^{-1})\begin{bmatrix} u \\ d \end{bmatrix}\right\rangle_{L_2\times\mathbb{C}} \\ &= \left\langle v, A_1M_0(z)H_0^{-1}\left[-W_bR(b)\int_a^b M_0(b,\tau)u(\tau)\,d\tau + d\right]\right\rangle_{L_2} \\ &\quad + \left\langle v, A_1\int_a^z M_0(z,\tau)u(\tau)\,d\tau\right\rangle_{L_2}.\end{aligned}$$

The proof follows easily from this by using Fubini's theorem and making use of a change of variable. ∎

Proposition 4.23: *Assume all conditions in Theorem* 4.20 *are satisfied. Then, the subspace* $\mathcal{V}$ *is given by*

$$\mathcal{V} = \left\{ y \in H^1(a,b)^n \;\middle|\; W_b(Ry)(b) + W_a(Ry)(a) = 0 \right\},$$

and it is dense in $L_2(a,b)^n$. ♡

PROOF: First we show that the kernel of $(T_1T_0^{-1})^*$ is $\{0\}\times\mathbb{C}^n$. To do so, let $\left[\begin{smallmatrix} v \\ c \end{smallmatrix}\right]$ be such that $(T_1T_0^{-1})^*\left[\begin{smallmatrix} v \\ c \end{smallmatrix}\right] = 0$. This implies, from (4.50), that

$$\begin{aligned} 0 = &-M_0^T(b,z)\int_a^b R^T(b)W_b^TH_0^{-T}M_0^T(\tau)(A_1v)(\tau)\,d\tau + \int_z^b M_0^T(\tau,z)(A_1v)(\tau)\,d\tau \\ \Rightarrow\ &M_0^T(b,z)\int_a^b R^T(b)W_b^TH_0^{-T}M_0^T(\tau)(A_1v)(\tau)\,d\tau = \int_z^b M_0^T(\tau,z)(A_1v)(\tau)\,d\tau. \end{aligned} \tag{4.51}$$

Differentiating (with respect to z) the equation above yields

$$\begin{aligned} 0 = A_0^T(z)M_0^T(b,z)&\int_a^b R^T(b)W_b^TH_0^{-T}M_0^T(\tau)(A_1v)(\tau)\,d\tau \\ &- A_0^T(z)\int_z^b M_0^T(\tau,z)(A_1v)(\tau)\,d\tau + (A_1v)(z). \end{aligned}$$

Substituting (4.51) in the equation above gives

$$A_1 v(z) = 0, \quad \Rightarrow \quad v(z) = 0.$$

From this we conclude that $\ker(T_1 T_0^{-1})^* = \{0\} \times \mathbb{C}^n$. Now note that $(T_1 T_0^{-1})^* \left[\begin{smallmatrix} v \\ c \end{smallmatrix}\right] \in \ker(T_1 T_0^{-1})^*$ if

$$0 = -M_0^T(b,z) \int_a^b R^T(b) W_b^T H_0^{-T} M_0^T(\tau)(A_1 v)(\tau)\, d\tau + \int_z^b M_0^T(\tau, z)(A_1 v)(\tau)\, d\tau.$$

The argument presented above gives $v(z) = 0$. Hence we have $\ker\Big((T_1 T_0^{-1})^*\Big)^k = \ker(T_1 T_0^{-1})^* = \{0\} \times \mathbb{C}^n$ for $k \geq 1$. Thus, $y \in H^1(a,b)^n$ is in $\mathcal{V}$, see (4.45), if and only if

$$\begin{aligned} y \in \mathcal{V} \quad &\Longleftrightarrow \quad T_0 y \;\perp\; \{0\} \times \mathbb{C}^n \\ &\Longleftrightarrow \quad \left\langle T_0 y, \begin{bmatrix} 0 \\ I \end{bmatrix} \right\rangle_{L_2 \times \mathbb{C}^n} = 0 \\ &\Longleftrightarrow \quad W_b (Ry)(b) + W_a (Ry)(a) = 0. \end{aligned}$$

From this the result follows. The denseness follows from the denseness of the domain of the semigroup generator $\mathcal{A}_{\mathcal{L}}$, since $\mathcal{V} = D(R^{-1}\mathcal{L}\mathcal{A}_{\mathcal{L}})$ and $(R^{-1}\mathcal{L})^{-1}$ is invertible. ∎

Theorem 4.24: *Assume all conditions of Theorem* 4.20 *are satisfied and let* $\{y_\nu^s\}$ *be a canonical system of eigenvectors and associated vectors of* (4.21) *and* (4.23) *and hence of the linear pencil* $T(\lambda) = T_0 - \lambda T_1$. *Then the canonical system* $\{y_\nu^s\}$ *is complete in* $L_2(a,b)^n$*, i.e.,*

$$\overline{\operatorname{span}\{y_\nu^s\}} = L_2(a,b)^n.$$

PROOF: We want to approximate any function, say x, in $L_2(a,b)^n$ by a linear combination of $\{y_\nu^s\}$. First note, from Theorem 4.20, that $\overline{\operatorname{span}\{y_\nu^s\}} = \mathcal{V}$, this implies that for all $y \in \mathcal{V}$ and for all $\epsilon > 0$ there exist $N > 0$ and α_ν, $\nu = 1, \ldots, N$, such that

$$\left\| y - \sum_{\nu=0}^{N} \alpha_\nu\, y_\nu^s \right\|_{H^1(a,b)^n} < \frac{\epsilon}{2}.$$

Also, from Proposition 4.23, it is easy to see that $\mathcal{V}$ is dense in $L_2(a,b)^n$, and hence for $x \in L_2(a,b)^n$ there exist $y \in \mathcal{V}$ such that

$$\|x - y\| < \frac{\epsilon}{2}.$$

Altogether gives for $x \in L_2(a,b)^n$

$$\begin{aligned}\left\| x - \sum_{\nu=0}^{N} \alpha_\nu \, y_\nu^s \right\| &\leq \|x-y\| + \left\| y - \sum_{\nu=0}^{N} \alpha_\nu \, y_\nu^s \right\| \\ &\leq \|x-y\| + \left\| y - \sum_{\nu=0}^{N} \alpha_\nu \, y_\nu^s \right\|_{H^1(a,b)^n} \\ &\leq \epsilon,\end{aligned}$$

where we used the fact that $\|w\| \leq \|w\|_{H^1(a,b)^n}$ for all $w \in H^1(a,b)^n$. This shows that $\operatorname{span}\{y_\nu^s\}$ is dense in $L_2(a,b)^n$ and from this the result follows. ∎

Proposition 4.25: *Assume all conditions on Theorem* 4.20 *hold. Then, the subspace $\mathcal{W}$ is given by*

$$\mathcal{W} = L_2(a,b)^n \times \mathbb{C}^n. \tag{4.52}$$

♡

Proof: First we we study the kernel of the operator $T_0^{-1}T_1$. First observe from equation (4.48) that

$$\begin{aligned}(T_0^{-1}T_1 y)(z) = M_0(z) H_0^{-1} &\left[-W_b R(b) \int_a^b M_0(b,\tau)(A_1 y)(\tau)\, d\tau \right] \\ &+ \int_a^z M_0(z,\tau)(A_1 y)(\tau)\, d\tau.\end{aligned} \tag{4.53}$$

From this and since A_1 is invertible we immediately see that

$$\ker(T_0^{-1}T_1) = \ker(T_1) = \{0\}, \quad \text{and hence} \quad \ker(T_0^{-1}T_1)^2 = \{0\}.$$

Therefore, with respect to the inner product on $H^1(a,b)^n$, $\left(\ker(T_0^{-1}T_1)^2\right)^\perp = H^1(a,b)^n$. In particular, the subspace $\mathcal{W}$ appearing in Theorem 4.20 is

$$\mathcal{W} = T_0^{-*}\left(H^1(a,b)^n\right) = L_2(a,b)^n \times \mathbb{C}^n.$$

∎

From this we obtain the following proposition.

Proposition 4.26: *Assume all conditions of Theorem* 4.20 *hold. Then,* $(T_1\mathcal{V})^\perp = \{0\} \times \mathbb{C}^n$*, and consequently*

$$m_0 = \operatorname{codim}_{\mathcal{QW}}(\mathcal{QW} \cap \mathcal{Q}(T_1\mathcal{V})^\perp) = 0.$$

♡

PROOF: The first condition is easy to prove. Indeed, any $\left[\begin{smallmatrix} u \\ d \end{smallmatrix}\right] \in L_2(a,b)^n \times \mathbb{C}^n$ is in $(T_1\mathcal{V})^\perp$ if and only if for all $y \in \mathcal{V}$

$$\begin{aligned} 0 &= \left\langle \begin{bmatrix} u \\ d \end{bmatrix}, T_1 y \right\rangle_{L_2 \times \mathbb{C}} = \left\langle \begin{bmatrix} u \\ d \end{bmatrix}, \begin{bmatrix} A_1 y \\ 0 \end{bmatrix} \right\rangle_{L_2 \times \mathbb{C}} \\ &= \langle u, A_1 y \rangle_{L_2} = \langle A_1 u, y \rangle_{L_2}. \end{aligned}$$

Since $\mathcal{V}$ is dense in $L_2(a,b)^n$, see Proposition 4.23, it follows that $A_1 u = 0$ and consequently $u = 0$. The rest of the proof follows easily from this and (4.52), since $\mathcal{Q}\mathcal{W} = \mathbb{C}^n$ and $\mathcal{Q}(T_1\mathcal{V})^\perp = \mathbb{C}^n$. ■

Following Theorem 3.11 of [Tre00b] we obtain the following theorem.

Theorem 4.27: *Assume that the problem* (4.11) *and* (4.13) *is normal and that* $A_0,\ A_1 \in M_n(W^{2,\infty}(a,b))$. *Let* $\{y_\nu^s\}$ *be a canonical system of eigenvectors and associated vectors of* (4.21) *and* (4.23) *and hence of the linear pencil* $T(\lambda) = T_0 - \lambda T_1$. *Then* $\{y_\nu^s\}$ *is a Riesz basis with parentheses in the subspace* $\mathcal{V} \subset H^1(a,b)^n$ *described in 4.20 and a Riesz basis with parentheses in* $L_2(a,b)^n$.

Moreover, it is only necessary to combine in parentheses the eigenfunctions and associated functions corresponding to eigenvalues the distance between which is smaller than δ *where* $\delta > 0$ *may be chosen arbitrarily small. If all the eigenvalues satisfy*

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0,$$

the basis property hold without parentheses.

PROOF: The proof can be found in [Tre00b, §3]. Observe that there, the basis property in $L_2(a,b)^n$ has finite defect less or equal to m_0, see Theorem 4.20. But following Proposition 4.26, we have that $m_0 = 0$. ■

Corollary 4.28: *Assume that the problem* (4.11) *and* (4.13) *is normal and that* $\mathcal{L} \in M_n(W^{2,\infty}(a,b))$. *Let* $\{x_\nu^s\}$ *be a canonical system of eigenvectors and associated vectors of the operator* $\mathcal{A}_\mathcal{L}$ *given by* (4.10). *If the eigenvalues of* $\mathcal{A}_\mathcal{L}$ *satisfy*

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0,$$

then $\{x_\nu^s\}$ *is a Riesz basis in* $L_2(a,b)^n$. ♠

PROOF: This follows from Theorem 4.27 since $\mathcal{L} \in M_n(W^{2,\infty}(a,b))$ implies that $A_0,\ A_1 \in M_n(W^{2,\infty}(a,b))$ and $x_\nu^s = \mathcal{L}^{-1} R y_\nu^s$, see Lemma 4.15, where $\mathcal{L}^{-1}R$ is an invertible operator. ■

Example 4.29 (Continuation of Example 4.14) Consider the system described in Example 4.14. There we had that the eigenvalues are expressed by (4.41) for large λ. It is clear from there that the eigenvalues are separated, i.e., $\inf_{n\neq m}|\lambda_n - \lambda_m| > 0$. It thus follows that the system has the Riesz basis property, and hence, it satisfies the spectrum-determined growth condition provided $k_b\Big(1+\sqrt{(T\,\rho)(b)}\Big)\neq 0$ and $k_b\Big(1-\sqrt{(T\,\rho)(b)}\Big)\neq 0$, i.e., the system is normal. From this we can conclude exponential stability as follows. Note that for $|\lambda| > M_1$, we have from (4.41)

$$\mathrm{Re}\,\lambda_m = -\frac{1}{2\int_a^b\sqrt{\frac{\rho(\xi)}{T(\xi)}}\,d\xi}\ln\left|\frac{\sqrt{(T\,\rho)(b)}+1}{\sqrt{(T\,\rho)(b)}-1}\right| < -\delta < 0, \qquad \delta > 0.$$

Thus if $\mathcal{A}_{\mathcal{L}}$ does not have eigenvalues on the imaginary axis, the system will be exponentially stable. To check this, assume that $\lambda_0 \in i\mathbb{R}$ is an eigenvalue of $\mathcal{A}_{\mathcal{L}}$ with corresponding eigenvector v. Then, by using the boundary conditions we have

$$\begin{aligned}
0 = \mathrm{Re}\left(\lambda_0\langle v,v\rangle_{\mathcal{L}}\right) = \mathrm{Re}\,\langle\mathcal{A}_{\mathcal{L}}v,v\rangle_{\mathcal{L}} &= \left(\begin{bmatrix}\frac{1}{\rho(z)}v_1(z)\\(Tv_2)(z)\end{bmatrix}^T P_1\begin{bmatrix}\frac{1}{\rho(z)}v_1(z)\\(Tv_2)(z)\end{bmatrix}\right)_{z=a}^{z=b}\\
&= \begin{bmatrix}-(Tv_2)(b)\\(Tv_2)(b)\end{bmatrix}^T\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}-(Tv_2)(b)\\(Tv_2)(b)\end{bmatrix} - \begin{bmatrix}\frac{v_1(a)}{\rho(a)}\\0\end{bmatrix}^T\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\frac{v_1(a)}{\rho(a)}\\0\end{bmatrix}\\
&= -\,2(Tv_2)^2(b),
\end{aligned} \tag{4.54}$$

which implies, see (4.36), that $(Tv_2)(b) = (\rho^{-1}v_1)(b) = 0$. This in turn, implies by Holmgren's Theorem, that $v(z) = 0$, which is a contradiction since it is an eigenvector. Thus, there are no eigenvalues on the imaginary axis, and we conclude that the system is exponentially stable. $*$

Example 4.30 (Timoshenko beam with boundary damping) In this example we study a linear system of Timoshenko type beam equations with boundary feedback. In [XF02] the authors prove that the system of eigenfunctions and associated eigenfunctions forms a Riesz basis in the state space. Here we prove the same result using our approach. This can be consider as a continuation of Example 3.27 on page 81. The model together with the port-variables are described in Example 2.19 on page 42.

The Riesz basis property of the Timoshenko beam under this boundary conditions was only proved recently by Xu and Feng in [XF02] where they dedicate that paper to prove that result. Here we prove the same result using our approach. First we find the operator (matrix) R that diagonalizes the matrix $(\mathcal{L}P_1)^{-1}$ and hence giving the eigenvalue problem (4.23). For simplicity, we assume that the beam is homogeneous and inextensible as it is done in [XF02].

This, in turn, implies that $\mathcal{L}$ is a constant matrix, simplifying the expression of A_0, see (4.21). This R is given by (for simplicity we denote EI by E)

$$R = \begin{bmatrix} \sqrt{K\rho} & 0 & -\sqrt{K\rho} & 0 \\ 1 & 0 & 1 & 0 \\ 0 & \sqrt{E\,I_\rho} & 0 & -\sqrt{E\,I_\rho} \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Using equation (4.21) this gives $A_1 = \operatorname{diag}\{\sqrt{\frac{\rho}{K}}, \sqrt{\frac{I_\rho}{E}}, -\sqrt{\frac{\rho}{K}}, -\sqrt{\frac{I_\rho}{E}}\}$ and

$$A_0 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2}\sqrt{\frac{K\rho}{E\,I_\rho}} & 0 & \frac{1}{2}\sqrt{\frac{K\rho}{E\,I_\rho}} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2}\sqrt{\frac{K\rho}{E\,I_\rho}} & 0 & -\frac{1}{2}\sqrt{\frac{K\rho}{E\,I_\rho}} & 0 \end{bmatrix}.$$

If $\frac{\rho}{K} \neq \frac{I_\rho}{E}$ then the block diagonal matrices appearing in (4.24) have dimension 1. If otherwise $\frac{\rho}{K} = \frac{I_\rho}{E}$ then the block diagonal elements have dimension 2. In the first case, it is easy to see that $\Psi_0 = I$, since Ψ_0 is the fundamental matrix of $\Psi'_{0,\nu\nu} - A_{0,\nu\nu}\Psi_{0,\nu\nu} = 0$, see (4.28), and the diagonal elements of A_0, i.e., $A_{0,\nu\nu}$, are zero. In [XF02] the authors assumed $\frac{\rho}{K} \neq \frac{I_\rho}{E}$, and so we follow the same assumption, which implies $\Psi_0 = I$. From the selection of A_1 we have that the corresponding exponential matrix $E(z,\lambda)$ is given by

$$E(z,\lambda) = \operatorname{diag}\left\{ e^{\lambda\sqrt{\frac{\rho}{K}}(z-a)},\ e^{\lambda\sqrt{\frac{I_\rho}{E}}(z-a)},\ e^{-\lambda\sqrt{\frac{\rho}{K}}(z-a)},\ e^{-\lambda\sqrt{\frac{I_\rho}{E}}(z-a)} \right\}.$$

Note that the set $\mathcal{E}$ appearing in Theorem 4.10, in this case, is given by (noting that $R_1(b) = \sqrt{\frac{\rho}{K}}(b-a)$, $R_2(b) = \sqrt{\frac{I_\rho}{E}}(b-a)$, $R_3(b) = -\sqrt{\frac{\rho}{K}}(b-a)$ and $R_4(b) = -\sqrt{\frac{I_\rho}{E}}(b-a)$, see (4.25))

$$\begin{aligned} \mathcal{E} &= \left\{ \sum_{j=1}^{4} R_j(b),\ \sum_{j=1}^{3} R_j(b),\ \sum_{j=1}^{2} R_j(b),\ \sum_{j=2}^{4} R_j(b), \ldots,\ R_1(b), R_2(b),\ R_3(b),\ R_4(b) \right\} \\ &= \left\{ \left(\sqrt{\frac{\rho}{K}} + \sqrt{\frac{I_\rho}{E}}\right)(b-a),\ \ldots,\ 0,\ \ldots,\ -\left(\sqrt{\frac{\rho}{K}} + \sqrt{\frac{I_\rho}{E}}\right)(b-a) \right\}, \end{aligned} \tag{4.55}$$

where $\mathcal{E}$ has been organized in descending order. Recall that the matrix W that determines the boundary conditions (3.15) is given in (3.16) (with α_i, $i = \{1,2\}$, the given gain feedback constants). Thus, the matrices W_b and W_a in (4.10c) are

$$W_b = \sqrt{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 1 & \alpha_2 \end{bmatrix}, \quad W_a = \sqrt{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From (4.33) we then get

$$\begin{aligned}\det\Big(&W_bR(b)\Psi_0(b)E(b,\lambda)+W_aR(b)\Big)\\&=4\left((\sqrt{K\rho}+\alpha_1)\,\mathrm{e}^{\lambda\sqrt{\frac{\rho}{K}}(b-a)}+(\sqrt{K\rho}-\alpha_1)\,\mathrm{e}^{-\lambda\sqrt{\frac{\rho}{K}}(b-a)}\right)\\&\qquad\left((\sqrt{E\,I_\rho}+\alpha_2)\,\mathrm{e}^{\lambda\sqrt{\frac{I_\rho}{E}}(b-a)}+(\sqrt{E\,I_\rho}-\alpha_2)\,\mathrm{e}^{-\lambda\sqrt{\frac{I_\rho}{E}}(b-a)}\right).\end{aligned} \tag{4.56}$$

Following Definition 4.12 and (4.55) we see that

$$w_M=\left(\sqrt{\frac{\rho}{K}}+\sqrt{\frac{I_\rho}{E}}\right)(b-a),\quad\text{and}\quad w_m=-\left(\sqrt{\frac{\rho}{K}}+\sqrt{\frac{I_\rho}{E}}\right)(b-a).$$

Thus, the problem will be normal if the respective coefficients of $\mathrm{e}^{\lambda w_M}$ and $\mathrm{e}^{\lambda w_m}$ in Δ are nonzero. But from equation (4.56), we can see it is equivalent to $(\sqrt{K\rho}+\alpha_1)(\sqrt{E\,I_\rho}+\alpha_2)$ and $(\sqrt{K\rho}-\alpha_1)(\sqrt{E\,I_\rho}-\alpha_2)$ being different than zero, or which is the same, that $\sqrt{K\rho}\neq\alpha_1$ and $\sqrt{E\,I_\rho}\neq\alpha_2$. This last condition is similar to the one appearing in [XF02]. Again from Rouché's theorem, we get that the roots of the characteristic determinant can be approximated by those of (4.56) (for λ large enough). These roots can be found explicitly, and are given by

$$\tilde\mu_m=\begin{cases}\frac{1}{2(b-a)}\sqrt{\frac{K}{\rho}}\ln\left|\frac{(\sqrt{K\rho}-\alpha_1)}{(\sqrt{K\rho}+\alpha_1)}\right|+i\frac{\pi m}{(b-a)}\sqrt{\frac{K}{\rho}} & \text{if } (\sqrt{K\rho}-\alpha_1)<0\\[2ex] \frac{1}{2(b-a)}\sqrt{\frac{K}{\rho}}\ln\left|\frac{(\sqrt{K\rho}-\alpha_1)}{(\sqrt{K\rho}+\alpha_1)}\right|+i\frac{\pi(1+2m)}{2(b-a)}\sqrt{\frac{K}{\rho}} & \text{if } (\sqrt{K\rho}-\alpha_1)>0\end{cases}$$

$$\tilde\nu_m=\begin{cases}\frac{1}{2(b-a)}\sqrt{\frac{E}{I_\rho}}\ln\left|\frac{(\sqrt{E\,I_\rho}-\alpha_2)}{(\sqrt{E\,I_\rho}+\alpha_2)}\right|+i\frac{\pi m}{(b-a)}\sqrt{\frac{E}{I_\rho}} & \text{if } (\sqrt{E\,I_\rho}-\alpha_2)<0\\[2ex] \frac{1}{2(b-a)}\sqrt{\frac{E}{I_\rho}}\ln\left|\frac{(\sqrt{E\,I_\rho}-\alpha_2)}{(\sqrt{E\,I_\rho}+\alpha_2)}\right|+i\frac{\pi(1+2m)}{2(b-a)}\sqrt{\frac{E}{I_\rho}} & \text{if } (\sqrt{E\,I_\rho}-\alpha_2)>0\end{cases}$$

with $m\in\mathbb{Z}$. Thus, by Rouché's theorem, the roots of (4.32) satisfy for $\lambda=\mu$ or $\lambda=\nu$ large enough

$$\mu_m=\tilde\mu_m+O(m^{-1}),\quad\text{and}\quad\nu_m=\tilde\nu_m+O(m^{-1}),\quad |m|>N_1,\ m\in\mathbb{Z}, \tag{4.57}$$

where N_1 is some sufficiently large positive integer. Since we assumed $\frac{\rho}{K}\neq\frac{I_\rho}{E}$, it is easy to see that the eigenvalues satisfy the gap condition in Corollary 4.28 and, thus, the system has the Riesz basis property. Hence, the spectrum-determined growth condition is satisfied, see [CZ95b]. Observe that the real part of $\tilde\mu_m$ and $\tilde\nu_m$ is always negative. It is also possible to check that this system does not have eigenvalues on the imaginary axis, see Example 3.27, and hence we can conclude exponential stability by the spectrum-determined growth condition. $*$

Remark 4.31. Even though the approach may seen complicated, it simplifies (when applicable) the application of the existing methods since the result presented here holds for class of systems rather than a particular case. For instance, the completeness of the generalized eigenfunctions is already proved here, and thus the number of steps needed to prove the Riesz basis property is reduced. Compare Example 4.30 with [XF02]. ♣

Chapter 5

Stability and Stabilization

In this chapter we study stability and stabilization properties of the energy preserving systems described in Chapter 2 and 3. Stability is a key property in most of control systems, and its validity is one of the first aims in the design of control systems. It is well-know that stability analysis of infinite-dimensional systems is more complicated in compared with the finite-dimensional case. In this chapter we try to provide some tools that can facilitate the stability analysis of a class of distributed parameter systems. Broadly speaking, in the infinite-dimensional case we can find two types of stability- asymptotic stability and exponential stability - in contrast with the final-dimensional case where these two types are the same.

We start by studying asymptotic stability and then we give some ideas that can help to check exponential stability. We mainly focus on the output and scattering energy preserving systems, since this are the most common (among the class studied) in applications.

5.1. Asymptotic stability

As mentioned earlier, in this section we provide some tools that facilitate the analysis of asymptotic stability properties for output and scattering energy preserving systems. We start with output energy preserving systems (see Section 3.4) and show that this class of systems are obtained when static feedback is used for impedance passive systems. Following this we present some results which could be useful when dealing with static feedback as well as dynamic feedback.

5.1.1. Output energy preserving systems

Recall from Definition 3.18 that output energy preserving systems satisfy the energy equality (3.11), i.e.,

$$\|x(t)\|_X^2 - \|x_0\|_X^2 = 2\int_0^t \langle u(\tau), y(\tau)\rangle_U \, d\tau - 2\int_0^t \langle \alpha y(\tau), y(\tau)\rangle_U \, d\tau,$$

where α is positive semi-definite. Next we show that this class of systems emerge when static feedback is used on impedance energy preserving systems.

Relation with impedance passive energy preserving systems

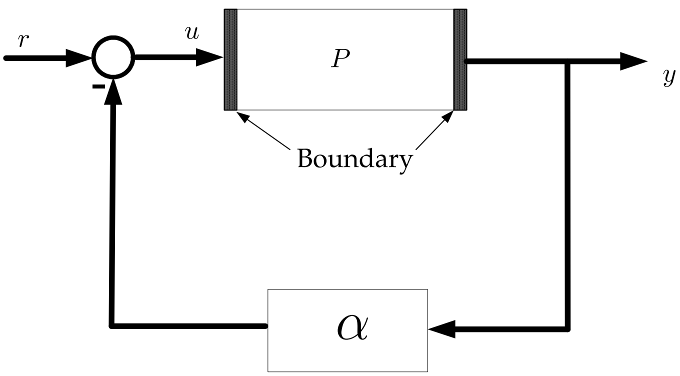

Figure 5.1.: Feedback loop.

Consider the feedback loop of Figure 5.1 where the plant is an impedance energy preserving system as described in Theorem 2.16. Observe, from the figure, that we have

$$u = r - \alpha y, \qquad (5.1)$$

where $r, u, y \in \mathbb{R}^{nN}$ and $\alpha \geq 0$ is a positive semi-definite matrix. We have that the plant is described in Theorem 2.14 where W and $\widetilde{W}$ satisfy the conditions in Theorem 2.16. In this case the feedback corresponds to a change in the boundary conditions, in which the closed-loop system is represented by

$$\begin{aligned} \dot{x}(t) &= \mathcal{J}\, x(t) \\ r(t) = (W + \alpha\, \widetilde{W}) \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} &= (\mathcal{B} + \alpha\, \mathcal{C})\, x(t) \\ \mathcal{C}\, x(t) &= y(t). \end{aligned} \tag{5.2}$$

It is easy to check that, in this case, $W_{\text{new}} = (W + \alpha\, \widetilde{W})$ satisfies $W_{\text{new}} \Sigma\, W_{\text{new}}^T = \alpha + \alpha^T$, which implies again by Theorem 2.14 that the closed-loop is a boundary control system since α is positive semi-definite, i.e., $\alpha \geq 0$, see [VZL+05b] for more details. Furthermore, following the same procedure used to prove (2.43)–(2.44), see [LZM05, p.19], we can show that if $x(0) \in D(\mathcal{J})$, and $(\mathcal{B} + \alpha\mathcal{C})x(0) = r(0)$, the closed-loop system satisfies

$$\frac{1}{2}\frac{d}{dt}\, \|x(t)\|_{\mathcal{L}}^2 = \langle r(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_U\,. \tag{5.3}$$

Hence, the closed-loop is output energy preserving. Recall from Theorem 2.16 that the semigroup generator of an impedance energy preserving system is skew-adjoint, and hence its eigenvalues lie on the imaginary axis. This implies that these systems are not asymptotically stable, that is why we need feedback to stabilize them. The next subsection presents some tools that help to check the stability property of the closed-loop system.

Asymptotic stability of output energy preserving systems

Recall, from Theorems 3.25 and 3.26, that we have related stability with the spectrum of the semigroup generator. That is, the system is asymptotically stable if the semigroup generator does not have eigenvalues on the imaginary axis. We emphasize that this result holds for the class of systems studied in Section 3.4 under Assumption 2.2. Thus, if we can conclude somehow that the semigroup operator does not have eigenvalues on the imaginary axis we are done. The next theorem provides a simple condition to check that.

Theorem 5.1: *Consider a BCS as described in Theorem 2.14. Assume it is output energy preserving, i.e.,* $\frac{1}{2}\frac{d}{dt}\, \|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_U$ *holds, and that* $\mathcal{L}$ *is a matrix whose entries are real analytic functions. Let* $\alpha \in \mathbb{R}^{nN\times nN}$. *If the matrix* W *satisfies*

$$W\,\Sigma\, W^T > 0, \quad \textit{or equivalently} \quad \alpha > 0,$$

then the system is asymptotically stable and the semigroup generator $\mathcal{A}_{\mathcal{L}}$ *does not have eigenvalues on the imaginary axis. Furthermore, all statements on Theorem 3.25 hold.*

Remark 5.2. A similar result (with $\mathcal{L} = I$) was proved in [VZL⁺05b] by using La Salle's invariance principle. ♣

PROOF (PROOF OF THEOREM 5.1): First notice from equation (3.12) that

$$P^{-1}_{W,\tilde{W}} = \begin{bmatrix} 2\alpha & I \\ I & 0 \end{bmatrix}.$$

This in turn implies from (2.44) that $2\alpha = W\,\Sigma\,W^T$ (see Remark 3.20) and thus $\alpha > 0$ (by the assumption on W). We know from Theorem 3.25 that we only need to prove that $\mathcal{A}_\mathcal{L}$ does not have eigenvalues on the imaginary axis. We prove this by contradiction. Assume that $\mathcal{A}_\mathcal{L}$ has an eigenvalue on the imaginary axis, say λ_0, with corresponding eigenvector $\phi_0 \in D(\mathcal{A}_\mathcal{L})$. Then, from Remark 2.15 we can deduce that

$$0 = \mathrm{Re}\,\lambda_0\,\langle \phi_0, \phi_0\rangle_\mathcal{L} = \mathrm{Re}\,\langle \mathcal{A}_\mathcal{L}\,\phi_0, \phi_0\rangle_\mathcal{L} = -\,\tau(\mathcal{L}\phi_0)^T R_{\mathrm{ext}}^T \widetilde{W}^T \alpha\,\underbrace{\widetilde{W} R_{\mathrm{ext}}\,\tau(\mathcal{L}\phi_0)}_{y}, \tag{5.4}$$

where we have used equation (3.13) and Definition 2.5. The equation above implies $\alpha\,\widetilde{W} R_{\mathrm{ext}}\,\tau(\mathcal{L}\phi_0)$ must be zero. Similarly, since $\phi_0 \in D(\mathcal{A}_\mathcal{L})$ implies $u = 0$ (see Remark 2.15) we must have $W R_{\mathrm{ext}}\,\tau(\mathcal{L}\phi_0) = 0$. In summary we have

$$\begin{bmatrix} W\,R_{\mathrm{ext}} \\ \alpha\,\widetilde{W} R_{\mathrm{ext}} \end{bmatrix} \tau(\mathcal{L}\phi_0) = 0.$$

Since α, $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$, and R_{ext} are nonsingular, the condition above implies that (see Definition 2.5)

$$\begin{bmatrix} (\mathcal{L}\phi_0)(b) \\ \vdots \\ \frac{d^{N-1}}{dz^{N-1}}(\mathcal{L}\phi_0)(b) \end{bmatrix} \text{ and } \begin{bmatrix} (\mathcal{L}\phi_0)(a) \\ \vdots \\ \frac{d^{N-1}}{dz^{N-1}}(\mathcal{L}\phi_0)(a) \end{bmatrix} \text{ are equal to zero.}$$

Thus, ϕ_0 is the solution of a PDE with *all* boundary variables set to zero for all $t \geq 0$. Therefore, we can use, by the assumption on $\mathcal{L}$, Holmgren's theorem (see Appendix A for the case $\mathcal{L}$ constant and [Joh49] for the general case) to conclude that $\phi_0 = 0$. Clearly this is a contradiction since ϕ_0 was assumed to be an eigenvector. Hence $\mathcal{A}_\mathcal{L}$ does not have eigenvalues on the imaginary axis. ■

Corollary 5.3: *Consider an impedance energy preserving BCS as described in Theorem 2.16. Then the feedback $u = -\alpha y$ with $\alpha > 0$ produces a (closed-loop) BCS that is output energy preserving, asymptotically stable, and exactly observable and controllable in infinite time.* ♠

Example 5.4 Consider an Euler-Bernoulli beam as described in Example 1.2. This model can be written as a system (2.1) by selecting the state variables

$$x_1 = \frac{\partial^2 w}{\partial z^2}: \quad \text{bending moment,}$$
$$x_2 = \rho\frac{\partial w}{\partial t}: \quad \text{transverse momentum distribution.}$$

Then the model of the beam can be rewritten as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\partial^2}{\partial z^2}}_{\mathcal{J}}\underbrace{\begin{bmatrix} EI\,x_1 \\ \frac{1}{\rho}\,x_2 \end{bmatrix}}_{\mathcal{L}\,x}. \tag{5.5}$$

From here we can see that the operator $\mathcal{J}$ is a second order differential operator of the form (2.2)–(2.3). It thus follows that $n = 2$, $\mathcal{L} = \mathrm{diag}\{EI,\ \rho^{-1}\} > 0$, and P_2 is a nonsingular skew-symmetric matrix. The energy of the system is known to be

$$\begin{aligned} E(t) =& \frac{1}{2}\int_a^b \left[\rho(z)\left|\frac{\partial w(z,t)}{\partial t}\right|^2 + EI(z)\left|\frac{\partial^2 w(z,t)}{\partial z^2}\right|^2\right] dz \\ =& \frac{1}{2}\int_a^b \left[\frac{1}{\rho(z)}|x_2(z,t)|^2 + EI(z)|x_1(z,t)|^2\right] dz = \frac{1}{2}\left\langle \mathcal{L}x, x\right\rangle. \end{aligned} \tag{5.6}$$

In this example $\mathcal{L}$ is assumed to satisfy the conditions on Theorem 5.1. Note that, in this case, we have that the matrix Q appearing in Stokes Theorem 2.1 is given by (note that $P_1 = 0$)

$$Q = \begin{bmatrix} P_1 & P_2 \\ -P_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{5.7}$$

Thus, the port-variables are (recall that $\partial_z = \frac{\partial}{\partial z}$)

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = R_{\text{ext}}\begin{bmatrix} (\mathcal{L}x)(b) \\ \frac{\partial(\mathcal{L}x)}{\partial z}(b) \\ (\mathcal{L}x)(a) \\ \frac{\partial(\mathcal{L}x)}{\partial z}(a) \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} \partial_z(\rho^{-1}x_2)(b) - \partial_z(\rho^{-1}x_2)(a) \\ -\partial_z(EIx_1)(b) + \partial_z(EIx_1)(a) \\ -(\rho^{-1}x_2)(b) + (\rho^{-1}x_2)(a) \\ (EIx_1)(b) - (EIx_1)(a) \\ (EIx_1)(b) + (EIx_1)(a) \\ (\rho^{-1}x_2)(b) + (\rho^{-1}x_2)(a) \\ \partial_z(EIx_1)(b) + \partial_z(EIx_1)(a) \\ \partial_z(\rho^{-1}x_2)(b) + \partial_z(\rho^{-1}x_2)(a) \end{bmatrix}. \tag{5.8}$$

As boundary conditions we set

$$(\rho^{-1}x_2)(a) = -c_1\partial_z(EIx_1)(a), \qquad \partial_z(\rho^{-1}x_2)(a) = c_2(EIx_1)(a),$$
$$(EIx_1)(b) = -k_1\partial_z(\rho^{-1}x_2)(b), \qquad \partial_z(EIx_1)(b) = k_2(\rho^{-1}x_2)(b),$$

where c_1, c_2, k_1 and k_2 are positive feedback constants. This corresponds to feedback (damping) being used at $z = a$ and $z = b$. The above boundary conditions can be obtained from the port-variables by selecting the W matrix

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & c_1 & 1 & 0 & 0 & 1 & c_1 & 0 \\ -1 & 0 & 0 & c_2 & -c_2 & 0 & 0 & 1 \\ k_1 & 0 & 0 & 1 & 1 & 0 & 0 & k_1 \\ 0 & -1 & k_2 & 0 & 0 & -k_2 & 1 & 0 \end{bmatrix},$$

which satisfies

$$W\Sigma W^T = 2 \underbrace{\begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & k_1 & 0 \\ 0 & 0 & 0 & k_2 \end{bmatrix}}_{\alpha}.$$

As output we choose

$$y = \begin{bmatrix} \partial_z(EIx_1)(a) \\ -(EIx_1)(a) \\ \partial_z(\rho^{-1}x_2)(b) \\ -(\rho^{-1}x_2)(b) \end{bmatrix}, \quad \text{with} \quad \widetilde{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Note that we have selected the variables which are used for feedback. It is now easy to check that this system is output energy preserving. Furthermore, it is asymptotically stable since the $\alpha > 0$, see Theorem 5.1. $*$

5.1.2. Dynamic boundary control of impedance energy preserving systems

In this subsection we generalize the class of static controllers described in the previous subsection. More precisely, we replace the static matrix α in Figure 5.1 with a matrix transfer function $\alpha(s)$ where $s \in \mathbb{C}$ is a complex variable. In this section we assume that the plant is an impedance energy preserving system, i.e.,

$$\dot{x}(t) = \mathcal{J}\mathcal{L}x(t) \tag{5.9a}$$

$$u(t) = \mathcal{B}\,x(t) = W \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} \tag{5.9b}$$

$$y(t) = \mathcal{C}\,x(t) = \widetilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, \tag{5.9c}$$

where W and $\widetilde{W}$ satisfy the conditions on Theorem 2.16.

The state space representation of the (finite-dimensional) controller is given by

$$\begin{aligned} \dot{v}(t) &= A_\alpha v(t) + B_\alpha y(t) \\ y_\alpha(t) &= C_\alpha v(t) + D_\alpha y(t) \end{aligned}, \tag{5.10}$$

where $v \in \mathbb{R}^m$ is the state of the minimal realization. The corresponding transfer function is denoted by $\alpha(s)$. In this way, equation (5.1) becomes

$$u = r - y_\alpha = r - C_\alpha v - D_\alpha \mathcal{C}\, x, \tag{5.11}$$

where (5.9c) was used. In this sense, the control law above is called a *dynamic boundary control*, in contrast to the control law given by (5.1) which describes a static relation. In this subsection the controller is assumed to be strictly positive real.

Definition 5.5 (Tao and Ioannou [TI88]). An $m \times m$ rational matrix $H(s)$ is said to be *positive real (PR)* if: i) all elements of $H(s)$ are analytic in the open right-half plane Re $(s) > 0$, ii) poles of any element of $H(s)$ on the jw-axis are distinct, and the associated residue matrix of $H(s)$ is ≥ 0, iii) $H(jw) + H^T(-jw) \geq 0\ \forall w$ which is not a pole of any element of $H(jw)$. A rational matrix $H(s)$ is *strictly positive real (SPR)* if $H(s - \epsilon)$ is positive real (PR) for some $\epsilon > 0$. ♣

Observe that a PR function may have poles on the imaginary axis, whereas all the poles of SPR functions are in the open left-half plane. The main advantage of PR or SPR functions is that they enable one to use a Lyapunov function, and hence to apply Lyapunov stability theory easily. The next lemma is an important tool in such stability analysis (see [TI88]).

Lemma 5.6 (KYP-Lemma): *Assume that the rational transfer matrix $H(s)$ has all its poles in Re $(s) < -\gamma$, where $\gamma > 0$ and (A, B, C, D) is a minimal realization of $H(s)$. Then $H(s - \gamma)$ is PR if and only if there exist matrices P, Q and K such that $P = P^T > 0$ and*

$$PA + A^T P = -QQ^T - 2\gamma P;\ PB = C^T - QK;\ K^T K = D + D^T. \tag{5.12}$$

♡

Next we find the closed-loop system. Let $x \in X$ be the state of the plant with X described by (2.33), $v \in \mathbb{R}^m$ the state of the controller, and $w = \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right]$. Using the feedback control law (5.11) and the fact that $u(t) = \mathcal{B}x(t)$, see (5.9b), we can see that the closed-loop system is described by

$$\begin{aligned} \dot{w}(t) &= \begin{bmatrix} \mathcal{J}\mathcal{L} & 0 \\ B_\alpha \mathcal{C} & A_\alpha \end{bmatrix} w(t) \\ r(t) &= (\mathcal{B} + D_\alpha\, \mathcal{C})\, x(t) + C_\alpha v(t) \\ y(t) &= \mathcal{C}\, x(t) \end{aligned}$$

or, which can be rewritten,

$$\begin{aligned} \dot{w}(t) &= \mathcal{J}_c\, w(t), \qquad w(0) \in \tilde{X} \\ r(t) &= \left[\begin{array}{cc} \mathcal{B} + D_\alpha\, \mathcal{C}, & C_\alpha \end{array}\right] w(t) \\ y(t) &= \left[\begin{array}{cc} \mathcal{C} & 0 \end{array}\right] w(t), \end{aligned} \tag{5.13}$$

where $\tilde{X} = X \times \mathbb{R}^m$ is the state space of the closed-loop system, $w = \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \tilde{X}$, and $\mathcal{J}_c : \tilde{X} \to \tilde{X}$ is a linear operator defined by

$$\mathcal{J}_c\, w = \left[\begin{array}{cc} \mathcal{J}\mathcal{L} & 0 \\ B_\alpha \mathcal{C} & A_\alpha \end{array}\right] \left[\begin{array}{c} x \\ v \end{array}\right] \quad \text{with} \quad D(\mathcal{J}_c) = \mathcal{L}^{-1} H^N(a,b;\mathbb{R}^n) \times \mathbb{R}^m. \tag{5.14}$$

The inner product on the space $\tilde{X}$ is defined as

$$\langle w_1, w_2 \rangle_{\tilde{X}} = \langle x_1, x_2 \rangle_{\mathcal{L}} + \frac{1}{2} v_1^T P v_2 + \frac{1}{2} v_2^T P v_1, \quad \forall\, w_i = \left[\begin{array}{c} x_i \\ v_i \end{array}\right] \in \tilde{X},\ i = \{1,2\}, \tag{5.15}$$

where P is the positive definite matrix found in Lemma 5.6.

Remark 5.7. It is know that the assumption on the controller being SPR is equivalent to being passive, see [ÇHvdSS03]. Also, it is possible to prove that the controller (5.10) can be represented as a finite-dimensional port-Hamiltonian system (with dissipation) with energy equal to $\frac{1}{2} v^T P v$. In fact, a finite-dimensional system described by the state space representation (5.10) has the so-called port-Hamiltonian form

$$\begin{aligned} \dot{v}(t) &= (J - R) L\, v(t) + (G - K)\, y(t) \\ y_\alpha(t) &= (G + K)^T L\, v(t) + (M + S)\, y(t), \end{aligned}$$

where J is a skew-symmetric $m \times m$ matrix, R is a symmetric $m \times m$ matrix, and L is an $m \times m$ matrix with $L = L^T \geq 0$. The Hamiltonian $H(v)$ (the energy of the system) is given by $H(v) = \frac{1}{2} v^T L\, v$. See, for instance [ÇHvdSS03]. Furthermore, G and K are $m \times n$ matrices, M is a skew-symmetric $n \times n$ matrix and S is a symmetric $n \times n$ matrix. Therefore, we can see the closed-loop system (5.13) as the interconnection (through the boundary) of an infinite-dimensional port-Hamiltonian system and a finite-dimensional port-Hamiltonian system. ♣

Next we need to check whether the closed-loop system (5.13) defines again a boundary control system as described in Definition 1.10.

Theorem 5.8: *Let the state of the open-loop system of Figure 5.1 satisfy* $\frac{1}{2} \frac{d}{dt} \|x(t)\|_{\mathcal{L}}^2 = u(t)^T y(t)$ *as described in Theorem 2.16 and let the controller*

$\alpha(s)$ be SPR. Then the system described by (5.13) *and* (5.14) *is a boundary control system on $\tilde{X} = X \times \mathbb{R}^m$. Furthermore, the operator $\mathcal{A}_c$ defined by*

$$\mathcal{A}_c\, w = \begin{bmatrix} \mathcal{J}\mathcal{L} & 0 \\ B_\alpha \mathcal{C} & A_\alpha \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \tag{5.16a}$$

with

$$D(\mathcal{A}_c) = \left\{ \begin{bmatrix} x \\ v \end{bmatrix} \in \begin{bmatrix} X \\ \mathbb{R}^m \end{bmatrix} \middle| \mathcal{L}x \in H^N(a,b;\mathbb{R}^n), \text{ and } \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \\ v \end{bmatrix} \in \ker \widetilde{W}_D \right\}, \tag{5.16b}$$

where

$$\widetilde{W}_D = \begin{bmatrix} (W + D_\alpha \widetilde{W}), & C_\alpha \end{bmatrix}, \tag{5.16c}$$

generates a contraction semigroup on $\tilde{X}$.

PROOF: First we need to prove that there exists an operator $\mathfrak{R} \in \mathcal{L}(U, \tilde{X})$ such that for all $r \in U$, $\mathfrak{R}r \in D(\mathcal{J}_c) \times \mathbb{R}^m$, and $\begin{bmatrix} \mathcal{B} + D_\alpha \mathcal{C}, & C_\alpha \end{bmatrix} \mathfrak{R}r = r$. From the proof of Theorem 4.5 of [LZM05] (see also the proof of Theorem 2.10 on page 32) we know that if the matrix $\widetilde{W}_D$ has full rank, then such operator $\mathfrak{R}$ exists. This follows by noticing that

$$(W + D_\alpha \widetilde{W}) = [I, \;\; D_\alpha] \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix},$$

where $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ is invertible. Hence $(W + D_\alpha \widetilde{W})$ has full-rank and so does $\widetilde{W}_D$.

Equations (5.16) follow easily from (5.13). Next we need to prove that $\mathcal{A}_c$ generates a semigroup. We will use the Lümer-Phillips theorem (see [Paz83]). First we prove that $\langle \mathcal{A}_c w, w \rangle \leq 0$. Let $w = \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(\mathcal{A}_c)$, then we have (recall that $\tilde{X}$ is a real Hilbert space)

$$\begin{aligned} \langle \mathcal{A}_c w, w \rangle_{\tilde{X}} &= \langle \mathcal{J}\mathcal{L}x, x \rangle_{\mathcal{L}} + \frac{1}{2}(A_\alpha v + B_\alpha y)^T P v + \frac{1}{2} v^T P (A_\alpha v + B_\alpha y) \\ &= \langle \mathcal{J}\mathcal{L}x, x \rangle_{\mathcal{L}} + \frac{1}{2} v^T (A_\alpha^T P + P A_\alpha) v + \frac{1}{2} y^T B_\alpha^T P v + \frac{1}{2} v^T P B_\alpha y. \end{aligned}$$

From equation (2.39) and Lemma 5.6 we obtain

$$\begin{aligned} \langle \mathcal{A}_c w, w \rangle_{\tilde{X}} = \frac{1}{2} \begin{bmatrix} f_\partial^T & e_\partial^T \end{bmatrix} \Sigma \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} + \frac{1}{2} v^T (-QQ^T - 2\gamma P) v \\ + \frac{1}{2} y^T (C_\alpha - K^T Q^T) v + \frac{1}{2} v^T (C_\alpha^T - QK) y. \end{aligned}$$

Using (2.49) together with (5.9b) and (5.9c) yields

$$\begin{aligned}\langle \mathcal{A}_c w, w\rangle_{\tilde{X}} = &\frac{1}{2}y^T u + \frac{1}{2}u^T y + \frac{1}{2}\, v^T(-QQ^T - 2\gamma P)v \\ &+ \frac{1}{2}y^T(C_\alpha - K^T Q^T)v + \frac{1}{2}\, v^T(C_\alpha^T - QK)y.\end{aligned}$$

Since $w = \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(\mathcal{A}_c)$ we have that $C_\alpha v = -(W + D_\alpha \widetilde{W}) \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$, see (5.16), and hence

$$\begin{aligned}\langle \mathcal{A}_c w, w\rangle_{\tilde{X}} = &\frac{1}{2}y^T u + \frac{1}{2}u^T y + \frac{1}{2}\, v^T(-QQ^T - 2\gamma P)v - \frac{1}{2}y^T(W + D_\alpha \widetilde{W}) \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] \\ &- \frac{1}{2}y^T K^T Q^T v - \frac{1}{2} \begin{bmatrix} f_\partial^T & e_\partial^T \end{bmatrix} (W^T + \widetilde{W}^T D_\alpha^T)y - \frac{1}{2}\, v^T QKy.\end{aligned}$$

Using again (5.9b) and (5.9c) gives, after simplification

$$\begin{aligned}\langle \mathcal{A}_c w, w\rangle_{\tilde{X}} = &\frac{1}{2}\, v^T(-QQ^T - 2\gamma P)v - \frac{1}{2}y^T D_\alpha y - \frac{1}{2}y^T D_\alpha^T y \\ &- \frac{1}{2}y^T K^T Q^T v - \frac{1}{2}\, v^T QKy\end{aligned}$$

and using again Lemma 5.6 yields

$$\langle \mathcal{A}_c w, w\rangle_{\tilde{X}} = -\gamma\, v^T P v - \frac{1}{2}(Ky + Q^T v)^T(Ky + Q^T v). \tag{5.17}$$

Since $\gamma > 0$ and P is positive definite it thus follows from the equation above that $\langle \mathcal{A}_c w, w\rangle_{\tilde{X}} \leq 0$.

Next we need to prove that the range of $(I - \mathcal{A}_c)$ is equal to $\tilde{X}$. In order to do so, we can show that for all $\left[\begin{smallmatrix} f \\ z \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathbb{R}^m \end{smallmatrix}\right] = \tilde{X}$ there exists $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(\mathcal{A}_c)$ such that

$$\begin{bmatrix} f \\ z \end{bmatrix} = \begin{bmatrix} (I - \mathcal{J}\mathcal{L})x \\ -B_\alpha \mathcal{C}x + (I - A_\alpha)v \end{bmatrix}. \tag{5.18}$$

Observe that since $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(\mathcal{A}_c)$ we must have (see (5.16), (5.9b) and (5.9c))

$$(\mathcal{B} + D_\alpha\, \mathcal{C})x + C_\alpha v = 0. \tag{5.19}$$

We need to solve (5.18) and (5.19) for $\left[\begin{smallmatrix} f \\ z \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathbb{R}^m \end{smallmatrix}\right]$ given. Recall that A_α is assumed to have only eigenvalues with negative real part, and hence $(I - A_\alpha)$ is a nonsingular matrix. Using the lower equation of (5.18) into (5.19) yields

$$\begin{aligned}&(\mathcal{B} + D_\alpha\, \mathcal{C})x + C_\alpha(I - A_\alpha)^{-1}z + C_\alpha(I - A_\alpha)^{-1}B_\alpha\, \mathcal{C}x = 0 \\ \Rightarrow &(\mathcal{B} + \alpha(1)\, \mathcal{C})x = -C_\alpha(I - A_\alpha)^{-1}z,\end{aligned} \tag{5.20}$$

where $\alpha(1) = C_\alpha(I - A_\alpha)^{-1}B_\alpha + D_\alpha$. Note that if we find an x that satisfies (5.20) and the upper equation in (5.18) we will be done. To do so, let $x = x_{\text{new}} + \tilde{\mathfrak{R}}\tilde{z}$

where $\tilde{\mathfrak{R}}$ is such that $(\mathcal{B} + \alpha(1)\,\mathcal{C})\,\tilde{\mathfrak{R}} = I$ (the existence of $\tilde{\mathfrak{R}}$ follows from the surjectivity of $(\mathcal{B} + \alpha(1)\,\mathcal{C})$, see Theorem 2.6 and [LZM04, pp.18-19] for details) and for simplicity let $\tilde{z} = -C_\alpha(I - A_\alpha)^{-1}z$. This used in (5.20) and the upper equation in (5.18) gives

$$(I - \mathcal{J}\mathcal{L})x_{\text{new}} = f - (I - \mathcal{J}\mathcal{L})\tilde{\mathfrak{R}}\tilde{z} \tag{5.21}$$

$$(\mathcal{B} + \alpha(1)\,\mathcal{C})x_{\text{new}} = 0 \;\Rightarrow\; (W + \alpha(1)\widetilde{W}) \begin{bmatrix} f_{\partial,\mathcal{L}x_{\text{new}}} \\ e_{\partial,\mathcal{L}x_{\text{new}}} \end{bmatrix} = 0. \tag{5.22}$$

Following Subsection 5.1.1 (see equation (5.2) and the paragraph after that) it is not difficult to see that if (5.22) holds, then $\mathcal{J}\mathcal{L}$ with domain

$$\left\{ x_{\text{new}} \in X \;\mid\; \mathcal{L}x_{\text{new}} \in H^N(a,b)^n, \begin{bmatrix} f_{\partial,\mathcal{L}x_{\text{new}}} \\ e_{\partial,\mathcal{L}x_{\text{new}}} \end{bmatrix} \in \ker(W + \alpha(1)\widetilde{W}) \right\} \tag{5.23}$$

generates a contraction semigroup. This implies that $(I - \mathcal{J}\mathcal{L})$ has an inverse and hence x_{new} exists. Thus, given $\left[\begin{smallmatrix} f \\ z \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathbb{R}^m \end{smallmatrix}\right]$ we can find $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(\mathcal{A}_c)$ such that (5.18) and (5.19) holds. ∎

Recall that in the previous section the compactness of the resolvent of the semigroup generator played an important role in the proof of asymptotic stability of Theorem 5.1. Next we prove that in the case of dynamic feedback the closed-loop system still have a compact resolvent, provided that the semigroup generator of the open-loop system has the same property.

Theorem 5.9: *Consider the feedback system of Figure* 5.1 *where the plant satisfies* $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = u(t)^T y(t)$ *as described in Theorem 2.16 and let the controller* $\alpha(s)$ *be SPR. Then the operator* $\mathcal{A}_c$ *described in Theorem* 5.8 *has compact resolvent.*

PROOF: Denote by A the operator $A = \mathcal{J}\mathcal{L}$ with boundary conditions (5.22), i.e., with domain (5.23). This clearly has compact resolvent by Theorem 2.28. Below we prove that $(\lambda I - \mathcal{A}_c)^{-1}$ is compact for $\lambda = 1$, which implies by Theorem 6.29 of [Kat95, Th.6.29, ch3] that $(\lambda I - \mathcal{A}_c)^{-1}$ is compact for all $\lambda \in \rho(\mathcal{A}_c)$.

We apply Theorem 8.1-3 of [Kre89], which states that an operator is compact if and only if it maps every bounded sequence onto a sequence which has a convergent subsequence. First we find the inverse of $(\lambda I - \mathcal{A}_c)$ for $\lambda = 1$ by following the same procedure used to find (5.18)–(5.22). We know that this inverse exists since $\mathcal{A}_c$ generates a contraction semigroup. From (5.21)–(5.22) it is easy to see, in this case, that $x_{\text{new}} \in D(A)$ and $x_{\text{new}} = (I - A)^{-1}f - (I - A)^{-1}(I - \mathcal{J}\mathcal{L})\tilde{\mathfrak{R}}\tilde{z}$, where $\tilde{z} = -C_\alpha(I - A)^{-1}z$. Since $x = x_{\text{new}} + \tilde{\mathfrak{R}}\tilde{z}$ we obtain that

$$x = (I - A)^{-1}f - (I - A)^{-1}(I - \mathcal{J}\mathcal{L})\tilde{\mathfrak{R}}\tilde{z} + \tilde{\mathfrak{R}}\tilde{z}, \tag{5.24}$$

and from the lower equation of (5.18) we get

$$v = (I - A_\alpha)^{-1} B_\alpha \mathcal{C} x + (I - A_\alpha)^{-1} z. \tag{5.25}$$

Thus equations (5.24) and (5.25) describe $(I - \mathcal{A}_c)^{-1}$, see (5.18). Let $\{k_n\} = \{\left[\begin{smallmatrix} f_n \\ z_n \end{smallmatrix}\right]\} \in \tilde{X} = X \times \mathbb{R}^m$ be any bounded sequence in $\tilde{X}$ and let $w_n = \left[\begin{smallmatrix} x_n \\ v_n \end{smallmatrix}\right] \in D(\mathcal{A}_c)$ be such that $w_n = (I - \mathcal{A}_c)^{-1} k_n$. By Theorem 5.8 we know that $\mathcal{A}_c$ generates a contraction semigroup and by the Hille-Yosida theorem it follows that $\left\|(\lambda I - \mathcal{A}_c)^{-1}\right\| \leq \frac{1}{\lambda}$ for $\lambda > 0$. Hence the sequence $\{w_n\}$ is bounded too. Since we know that $(I - A)^{-1}$ is compact and that $\mathcal{J}\mathcal{L}\tilde{\mathfrak{R}}$ is bounded (see Definition 1.10), we have that $\{x_n\}$ has a convergent subsequence, see (5.24). Also, since $\{v_n\}$ is bounded and belongs to a finite dimensional subspace of $\tilde{X}$, it follows that $\{v_n\}$ has another convergent subsequence. Hence, we can conclude that $w_n = \left[\begin{smallmatrix} x_n \\ v_n \end{smallmatrix}\right]$ has a convergent subsequence and therefore $(\lambda I - \mathcal{A}_c)^{-1}$ is compact for $\lambda = 1$ (and hence for all $\lambda \in \rho(\mathcal{A}_c)$). ■

Next we give an asymptotic stability result.

Theorem 5.10: *Consider the feedback system of Figure* 5.1 *where the plant satisfies* $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = u(t)^T y(t)$ *as described in Theorem 2.16 with* $\mathcal{L}$ *a matrix whose entries are real analytic functions. Let the controller* $\alpha(s)$ *be SPR. Then the system described by* (5.13)–(5.14) *with* $r = 0$*, is globally asymptotically stable. That is for any* $w(0) \in \tilde{X}$*, the unique (classical or weak) solution* $w(t) = T_c(t)w(0)$ *of* (5.13) *asymptotically approaches to zero, i.e.,* $\lim_{t\to\infty} \|w(t)\|_{\tilde{X}} = 0$.

PROOF: First we prove this for $w(0) \in D(\mathcal{A}_c)$. By Theorem 5.8 we know that $\mathcal{A}_c$ generates a contraction semigroup, say T_c. In this case we have that $w(t) = T_c(t)w(0) \in D(\mathcal{A}_c)$ for all $t \geq 0$, see Theorem 2.1.10 of [CZ95b]. Define the energy function

$$E_c(t) = \frac{1}{2}\|w(t)\|^2_{\tilde{X}} = \frac{1}{2}\langle w(t), w(t)\rangle_{\tilde{X}}. \tag{5.26}$$

Since $w(0) \in D(\mathcal{A}_c)$ we have that $w(t)$ is differentiable, see [CZ95b, §2.1]. By differentiating the equation above and using (5.13) and (5.17) we obtain (recall that $\tilde{X}$ is a real Hilbert space)

$$\begin{aligned} \dot{E}_c(t) &= \langle \dot{w}(t), w(t)\rangle_{\tilde{X}} = \langle \mathcal{A}_c w(t), w(t)\rangle_{\tilde{X}} \\ &= -\gamma\, v(t)^T P v(t) - \frac{1}{2}(Ky(t) + Q^T v(t))^T (Ky(t) + Q^T v(t)), \end{aligned} \tag{5.27}$$

where $\gamma > 0$ and P is positive definite. Since $(\lambda I - \mathcal{A}_c)^{-1}$ is compact (see Theorem 5.9) and $T_c(t)$ is a contraction, it follows from LaSalle's invariance principle (see Theorem 3.64 of [LGM99]) that all solutions of (5.13) asymptotically tend to the maximal invariant set of

$$\mathcal{O}_c = \{\tilde{x} \in \tilde{X} \mid \dot{E}_c(t) = 0\}. \tag{5.28}$$

Let $\mathcal{E}$ be the largest invariant subset of $\mathcal{O}_c$. Next we show that $\mathcal{E} = \{0\}$. The condition $\dot{E}_c(t) = 0$ implies, from (5.27), that $v(t) = 0$; and hence, $\dot{v}(t) = 0$ as well. Then by (5.10) we must have that $B_\alpha y(t) = 0$. Since $\alpha(s)$ is SPR, we have that $\alpha(jw) + \alpha^T(-jw) > 0$. This implies that if $y(t) \neq 0$

$$\begin{aligned} & y^T(t)[\alpha(jw) + \alpha^T(-jw)]y(t) > 0 \\ \Rightarrow \quad & y^T(t)[D_\alpha + D_\alpha^T]y(t) > 0 \\ \Rightarrow \quad & y^T(t)K^T K\, y(t) > 0. \end{aligned}$$

In the second step the facts $\alpha(jw) = C_\alpha(jw - A_\alpha)^{-1}B_\alpha + D_\alpha$ and $B_\alpha y(t) = 0$ were used, and in the third step we used (5.12). Since $v(t) = 0$ and $K^T K > 0$ it follows from (5.27) that $y(t) = 0$, and hence by (5.10) we also obtain $y_\alpha(t) = 0$.

Therefore from (5.13) and (5.16) it follows that the invariant solution of (5.13) in $\mathcal{O}_c$ reduces to the invariant solution of a PDE with all boundary variables set to zero. It thus follows from Holmgren's Theorem, see [Joh49], that $x(t) = 0$. Hence the asymptotic stability.

The same statement holds for $w(0) \in \tilde{X}$ by using a simple denseness argument, see [LGM99, p.270]. ■

Remark 5.11. Observe that the proof above is equivalent to prove that the semigroup generator does not have eigenvalues on the imaginary axis. ♣

Corollary 5.12: *Consider the feedback system of Figure* 5.1 *where the plant satisfies* $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = u(t)^T y(t)$ *as described in Theorem 2.16 and let the controller* $\alpha(s)$ *be SPR. If the operator* $\mathcal{A}_c$ *on Theorem 5.8 does not have eigenvalues on the imaginary axis, then the system described by* (5.13)–(5.14) *with* $r = 0$*, is globally asymptotically stable.* ♠

PROOF: Following Remark 5.11 it is easy to see that if $\mathcal{A}_c$ does not have eigenvalues on the imaginary axis then the closed-loop is asymptotically stable. Indeed, in this case we have that $\mathcal{A}_c$ has a compact resolvent, see Theorem 5.9, and since it generates a contraction semigroup, it follows that the conditions on the theorem of Arendt and Batty [AB88] are satisfied. ■

5.1.3. Scattering energy preserving systems

In this subsection we study the asymptotic stability of the class of scattering energy preserving systems described in Section 3.3. Here we show that this class of systems is always asymptotically stable. In fact, in many situations, this class is also exponentially stable, see Section 5.2.

Theorem 5.13: *Consider a BCS as described in Theorem* 2.14. *Assume it is scattering energy preserving, i.e.,* $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|_U^2 - \|y(t)\|_U^2$ *holds (see Theorem* 2.17*). Then the system is asymptotically stable and the semigroup generator* $\mathcal{A}_\mathcal{L}$ *does not have eigenvalues on the imaginary axis. Furthermore, all statements on Theorem 3.17 hold.*

PROOF: This follows easily by noting that $W\Sigma W^T = \frac{1}{2}I$ (see the proof of Theorem 2.17). Then by using the same arguments of the proof of Theorem 5.1 the result follows. ∎

5.2. Exponential stability

In this section we present results that can help to prove whether a system is exponentially stable. In part of this section we use some of the results presented in Chapter 4 and hence we mainly deal with the case $N = 1$, i.e., $\mathcal{J}\mathcal{L}$ is a first order differential operator. The proof is based on the following well-known result.

Theorem 5.14: *Let* $T(t)$ *be a uniformly bounded* C_0*-semigroup on a Hilbert space* H *with generator* A*. Then* $T(t)$ *is exponentially stable if and only if* $i\mathbb{R} \subset \rho(A)$ *and*

$$M_0 := \sup_{w\in\mathbb{R}} \left\|(iw - A)^{-1}\right\| < \infty.$$

This is a fundamental and well-documented result in the literature on strongly continuous semigroups. This result is also well known to the control theory community as the Huang Theorem (see [Hua85]) or the Gearhart-Prüss Theorem (see [Gea78]).

The following two theorems relate the results presented in Chapter 4 with the ones of the previous section. They also give more than the exponential stability itself, they determine further properties of the system.

Theorem 5.15: *Consider the operator* $\mathcal{A}_\mathcal{L}$ *described in* (4.10) *with* $\mathcal{L} \in M_n(W^{2,\infty}(a,b))$*. Assume that its corresponding eigenvalue problem* (4.11) *and* (4.13) *is normal. If the eigenvalues of* $\mathcal{A}_\mathcal{L}$ *satisfy*

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0,$$

then the system satisfies the spectrum-determined growth condition. In addition, if $i\mathbb{R} \subset \rho(\mathcal{A}_\mathcal{L})$ and the asymptotic expansion of the roots of (4.32) *satisfy* $\text{Re}\,\lambda < -\delta < 0$*, with δ a positive constant, then the system* (4.8) *is exponentially stable.*

PROOF: Since $\inf_{n\neq m} |\lambda_n - \lambda_m| > 0$ holds, we have from Corollary 4.28 that the system has the Riesz basis property, and thus the the spectrum-determined growth condition is satisfied, see [CZ95b]. By assumption $\mathcal{A}_\mathcal{L}$ does not have eigenvalues on the imaginary axis and there are no accumulation points on the imaginary axis at infinity. Thus, the statement on stability follows by using the spectrum-determined growth condition, see [CZ95b]. ∎

From this we immediately obtain the following result.

Theorem 5.16: *Consider the operator $\mathcal{A}_\mathcal{L}$ described in* (4.10) *with $\mathcal{L} \in M_n(W^{2,\infty}(a,b))$. Assume that its corresponding eigenvalue problem* (4.11) *and* (4.13) *is normal with eigenvalues satisfying*

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0.$$

If the asymptotic expansion of the roots of (4.32) *satisfy* $\text{Re}\,\lambda < -\delta < 0$*, with δ a positive constant, and the system is asymptotically stable, then the system* (4.8) *is exponentially stable.*

Some conditions to establish asymptotic stability for this class of systems, which are easy to check, are given in the previous section.

The following results provide some simple tools to check exponential stability.

Theorem 5.17: *Consider a BCS as described in Theorem* 2.14 *where the skew-symmetric operator $\mathcal{J}\mathcal{L}$ is given by (with P_1 symmetric and nonsingular)*

$$\mathcal{J}\mathcal{L}x = P_1 \frac{d}{dz}(\mathcal{L}x)(z) + P_0\,\mathcal{L}\,x, \tag{5.29}$$

with $\mathcal{L} \in M_n\big(L_2(a,b)\big)$ a coercive (multiplication) operator which is differentiable with bounded derivative. If $u(t) = 0$, for all $t \geq 0$, then the energy of the system $E(t) = \frac{1}{2}\|x(t)\|^2_\mathcal{L}$ satisfies for τ large enough

$$E(\tau) \leq c(\tau) \int_0^\tau \|(\mathcal{L}x)(t,b)\|^2_\mathbb{R}\, dt, \quad \text{and} \quad E(\tau) \leq c(\tau) \int_0^\tau \|(\mathcal{L}x)(t,a)\|^2_\mathbb{R}\, dt \tag{5.30}$$

where c is a positive constant that only depends on τ. If, in addition, the energy of the system (when $u(t) = 0$, for all $t \geq 0$) satisfies

$$\frac{1}{2}\frac{d}{dt}\left\|x(t)\right\|_{\mathcal{L}}^{2} = -\left\langle \alpha y(t), y(t)\right\rangle_{U} \tag{5.31}$$

with α positive definite, then the system is exponentially stable.

Remark 5.18. Note that the conditions on Theorem 2.14 have to be satisfied. In particular, that $\mathcal{A}_{\mathcal{L}}$ generates a *contraction* semigroup. ♣

PROOF (PROOF OF THEOREM 5.17): Recall that the energy of the system is given by

$$E(t) = \frac{1}{2}\left\langle x(t,z), \mathcal{L}(z)\, x(t,z)\right\rangle = \frac{1}{2}\int_{a}^{b} x^{T}(t,z)\mathcal{L}(z)\, x(t,z)\, dz. \tag{5.32}$$

We start by proving the estimates (5.30). To do so we employ the idea used by Cox and Zuazua in [CZ95a]. We define the (positive) function

$$F(z) = \int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^{T}(t,z)\mathcal{L}(z)x(t,z)\, dt, \tag{5.33}$$

where $\gamma > 0$, $\tau > 2\gamma(b-a)$, and $0 \leq t \leq \tau$. It thus follows that

$$\begin{aligned} F'(z) = &\int_{\gamma(b-z)}^{\tau-\gamma(b-z)} \left(\frac{\partial}{\partial z}x(t,z)\right)^{T} \mathcal{L}(z)x(t,z)\, dt \\ &+ \int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^{T}(t,z)\frac{\partial}{\partial z}(\mathcal{L}(z)x(t,z))\, dt + \gamma x^{T}(\gamma(b-z),z)\mathcal{L}(z)x(\gamma(b-z),z) \\ &+ \gamma x^{T}(\tau-\gamma(b-z),z)\mathcal{L}(z)x(\tau-\gamma(b-z),z). \end{aligned}$$

Since P_1 is nonsingular and $\frac{\partial x}{\partial t} = P_1\frac{d\mathcal{L}x}{dz} + P_0\mathcal{L}x$ we obtain (for simplicity we omit the dependence on z and t)

$$\begin{aligned} F'(z) = &\int_{\gamma(b-z)}^{\tau-\gamma(b-z)} \left[\mathcal{L}^{-1}\left(P_1^{-1}\frac{\partial x}{\partial t} - \mathcal{L}'x - P_1^{-1}P_0\mathcal{L}x\right)\right]^{T} \mathcal{L}x\, dt \\ &+ \int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^{T}P_1^{-1}\left(\frac{\partial x}{\partial t} - P_0\mathcal{L}x\right) dt + \gamma x^{T}(\gamma(b-z),z)\mathcal{L}(z)x(\gamma(b-z),z) \\ &+ \gamma x^{T}(\tau-\gamma(b-z),z)\mathcal{L}(z)x(\tau-\gamma(b-z),z) \end{aligned}$$

$$
\begin{aligned}
&= x^T(t,z)P_1^{-1}x(t,z)\big|_{t=\gamma(b-z)}^{t=\tau-\gamma(b-z)} - \int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^T\left(\mathcal{L}P_0^T P_1^{-1} + P_1^{-1}P_0\mathcal{L}\right) x\,dt \\
&\quad - \int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^T\mathcal{L}'\,x\,dt + \gamma x^T(\tau-\gamma(b-z),z)\mathcal{L}(z)x(\tau-\gamma(b-z),z) \\
&\quad + \gamma x^T(\gamma(b-z),z)\mathcal{L}(z)x(\gamma(b-z),z),
\end{aligned}
$$

where we used $P_1^T = P_1$, $\mathcal{L}^T = \mathcal{L}$. By simplifying the equation above one obtains

$$
\begin{aligned}
F'(z) = &-\int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^T\mathcal{L}'\,x\,dt - \int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^T\left(\mathcal{L}P_0^T P_1^{-1} + P_1^{-1}P_0\mathcal{L}\right) x\,dt \\
&+ x^T(\tau-\gamma(b-z),z)\left[P_1^{-1} + \gamma\mathcal{L}(z)\right]x(\tau-\gamma(b-z),z) \\
&+ x^T(\gamma(b-z),z)\left[-P_1^{-1} + \gamma\mathcal{L}(z)\right]\mathcal{L}(z)x(\gamma(b-z),z).
\end{aligned}
$$

By choosing γ large enough, i.e., by choosing τ large, we get that $P_1^{-1} + \gamma\mathcal{L}$ and $-P_1^{-1} + \gamma\mathcal{L}$ are positive definite. This in turn implies that (for τ large enough)

$$
F'(z) \geq -\int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^T\mathcal{L}'\,x\,dt - \int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^T\left(\mathcal{L}P_0^T P_1^{-1} + P_1^{-1}P_0\mathcal{L}\right) x\,dt.
$$

Since P_1 and P_0 are constant matrices and, by assumption, $\mathcal{L}'(z)$ is bounded, i.e., $x^T\mathcal{L}'(z)\,x \leq c\,x^T\mathcal{L}(z)\,x$, we get

$$
F'(z) \geq -\kappa\int_{\gamma(b-z)}^{\tau-\gamma(b-z)} x^T(t,z)\mathcal{L}(z)x(t,z)\,dt = -\kappa\,F(z),
$$

where κ is a positive constant and we used (5.33). Thus we have $\frac{F'(z)}{F(z)} \geq -\kappa$, which in turn implies (for τ large enough)

$$
\begin{aligned}
&\int_{z_1}^{z_2}\frac{F'(z)}{F(z)}\,dz \geq -\kappa\int_{z_1}^{z_2} dz, \qquad \text{for } z_2 \geq z_1, \\
\Rightarrow\quad & \ln\big(F(z_2)\big) - \ln\big(F(z_1)\big) \geq -\kappa\,(z_2 - z_1) \\
\Rightarrow\quad & F(z_2) \geq F(z_1)\,\mathrm{e}^{-\kappa\,(z_2-z_1)} \\
\Rightarrow\quad & F(b) \geq F(z)\,\mathrm{e}^{-\kappa\,(b-a)} \qquad \text{for } z \in [a,b].
\end{aligned}
\tag{5.34}
$$

Since we have that $E(t_2) \leq E(t_1)$ for any $t_2 \geq t_1$ (by the contraction property of the semigroup), we deduce that

$$
\begin{aligned}
\int_{\gamma(b-a)}^{\tau-\gamma(b-a)} E(t)\,dt &\geq E(\tau-\gamma(b-a))\int_{\gamma(b-a)}^{\tau-\gamma(b-a)} dt \\
&= (\tau - 2\gamma(b-a))E(\tau-\gamma(b-a)).
\end{aligned}
$$

Using the definition of $F(z)$ and $E(t)$, see (5.33) and (5.32), together with the equation above as well as the estimate (5.34), and the coercivity of $\mathcal{L}$ we obtain (for τ large enough)

$$\begin{aligned}
2(\tau - 2\gamma(b-a))E(\tau) &\leq 2(\tau - 2\gamma(b-a))E(\tau - \gamma(b-a)) \\
&\leq \int_a^b \int_{\gamma(b-a)}^{\tau-\gamma(b-a)} x^T(t,z)\mathcal{L}x(t,z)\,dt\,dz \\
&= \int_a^b F(z)\,dz \leq (b-a)F(b)\,\mathrm{e}^{\kappa\,(b-a)} = (b-a)\,\mathrm{e}^{\kappa\,(b-a)} \int_0^\tau x^T(t,b)\mathcal{L}x(t,b)\,dt \\
&\leq c_1 \int_0^\tau \|(\mathcal{L}x)(t,b)\|_{\mathbb{R}}^2\,dt,
\end{aligned}$$

where $c_1 = (b-a)\left\|\mathcal{L}^{-1}(b)\right\|\,\mathrm{e}^{\kappa\,(b-a)}$. This means that for τ large enough

$$E(\tau) \leq c_2(\tau)\int_0^\tau \|(\mathcal{L}x)(t,b)\|_{\mathbb{R}}^2\,dt, \qquad c_2 = \frac{(b-a)\left\|\mathcal{L}^{-1}(b)\right\|}{2(\tau - 2\gamma(b-a))}\,\mathrm{e}^{\kappa\,(b-a)}. \tag{5.35}$$

This proves the first estimate on (5.30). The other estimate follows by replacing $F(z)$ in the argument above by

$$\tilde{F}(z) = \int_{\gamma(z-a)}^{\tau-\gamma(z-a)} x^T(t,z)\mathcal{L}(z)x(t,z)\,dt.$$

Next we prove the exponential stability. If $u(t) = 0$, then we know by assumption, see (5.31), that the energy satisfies for some $\tau > 0$

$$E(0) - E(\tau) = \int_0^\tau \langle \alpha y(t), y(t)\rangle_U\,dt.$$

Now, observe that it is sufficient to prove the existence of some time $\tau > 0$ and some constant $c_0 > 0$ such that

$$E(\tau) \leq c_0 \int_0^\tau \langle \alpha y(t), y(t)\rangle_U\,dt \tag{5.36}$$

for all solutions of the system. Indeed, combining the two equations above we get

$$E(\tau) \leq \frac{c_0}{1+c_0}E(0).$$

From this we see that the semigroup $T(t)$ generated by $\mathcal{A}_\mathcal{L}$ satisfies $\|T(t)\| < 1$, from which we obtain exponential stability.

In order to find the estimate (5.36) we need to find a relation between $\left\|\mathcal{L}^{-1}(b)\right\|_{\mathbb{R}}$ (or $\left\|\mathcal{L}^{-1}(a)\right\|_{\mathbb{R}}$) and $\|y(t)\|_{\mathbb{R}}$. To do so, observe from Theorem 2.14 that (since

$u(t) = 0$)

$$\begin{bmatrix} 0 \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} W \\ \widetilde{W} \end{bmatrix} R_{\text{ext}}}_{M} \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix}.$$

Since $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ and R_{ext} are nonsingular it follows that M is invertible and, in particular, $\|M\,w\|^2_{\mathbb{R}} \geq \varepsilon\,\|w\|^2_{\mathbb{R}}$. Taking norms on both sides yields

$$\begin{aligned} \|y\|^2_{\mathbb{R}} &= \left\| M \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix} \right\|^2_{\mathbb{R}} \geq \varepsilon \left\| \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix} \right\|^2_{\mathbb{R}} \geq \varepsilon\,\|(\mathcal{L}x)(b)\|^2_{\mathbb{R}} \\ \Rightarrow \quad & \|(\mathcal{L}x)(b)\|^2_{\mathbb{R}} \leq \varepsilon^{-1}\,\|y\|^2_{\mathbb{R}}. \end{aligned}$$

This together with (5.35) and the coercivity of α gives the estimate (5.36). Hence, the system is exponentially stable. ■

The estimates (5.30) can be used to prove exponential stability in some cases when α is positive semi-definite, as the following corollary and examples show.

Corollary 5.19: *Consider a BCS as described in Theorem* 2.14 *where the operator* $\mathcal{J}\mathcal{L}$ *is given by (with* P_1 *symmetric and nonsingular)*

$$\mathcal{J}\mathcal{L}x = P_1 \frac{d}{dz}(\mathcal{L}x)(z) + P_0\,\mathcal{L}\,x, \tag{5.37}$$

with[1] $\mathcal{L} \in M_n\big(L_2(a,b)\big)$ *a coercive operator which is differentiable with bounded derivative. Assume that the energy of the system when* $u(t) = 0$ *for all* $t \geq 0$*, satisfies*

$$\frac{1}{2}\frac{d}{dt}\,\|x(t)\|^2_{\mathcal{L}} = -\,\langle \alpha y(t), y(t)\rangle_U\,,$$

where α *is a positive semi-definite matrix, i.e.,* $\alpha \geq 0$*. Then the system is exponentially stable if*

$$\text{either } \|(\mathcal{L}x)(b)\|^2_{\mathbb{R}} \leq k_1\,\langle \alpha\,y, y\rangle_U \text{ or } \|(\mathcal{L}x)(a)\|^2_{\mathbb{R}} \leq k_1\,\langle \alpha y, y\rangle_U \tag{5.38}$$

holds where k_1 *is a positive constant.* ♠

PROOF: It follows easily from the proof of Theorem 5.17. Indeed, by using the condition (5.38) in (5.30) we obtain the estimate (5.36). ■

Remark 5.20. Note that in the proof of Theorem 5.17 only the properties of P_1 were used and not of P_0. Thus, Theorem 5.17 and Corollary 5.19 also hold for any P_0 provided that $\mathcal{L}$ is coercive and the semigroup generated by $\mathcal{A}_{\mathcal{L}} = P_1 \frac{d}{dz}(\mathcal{L}x)(z) + P_0\,\mathcal{L}\,x$ is a contraction. ♣

[1] Note that the condition $\mathcal{L} \in M_n\big(L_2(a,b)\big)$ implies that $\mathcal{L}$ is a multiplication operator.

Example 5.21 Consider the wave equation of Example 4.14, which can be modeled by

$$\frac{\partial}{\partial t}\underbrace{\begin{bmatrix} p \\ q \end{bmatrix}}_{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial z} \begin{bmatrix} \frac{1}{\rho}p \\ Tq \end{bmatrix}, \quad t \geq 0, \tag{5.39}$$

$$T(a)\, q(a,t) = 0, \ \frac{1}{\rho(b)} p(b,t) + \alpha_1 T(b)\, q(b,t) = 0, \tag{5.40}$$

where $z \in [a,b]$. Here $T(z)$ and $\rho(z)$ are positive *smooth* functions and $\alpha_1 > 0$ is a constant feedback gain. We want to prove that the system is exponentially stable under this boundary conditions.

The port-variables are given in equation (4.37) where $e = \left[\begin{smallmatrix} e_p \\ e_q \end{smallmatrix}\right] = \left[\begin{smallmatrix} \frac{1}{\rho}p \\ Tq \end{smallmatrix}\right]$. To the boundary conditions there correspond a W matrix given by

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 1 \\ \alpha_1 & 1 & 1 & \alpha_1 \end{bmatrix} \quad \Rightarrow \quad W\Sigma W^T = 2 \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \alpha_1 \end{bmatrix}}_{\alpha}. \tag{5.41}$$

As output we select the variables

$$y = \begin{bmatrix} \frac{1}{\rho(a)} p(a,t) \\ T(b)\, q(b,t) \end{bmatrix}, \quad \text{with} \quad \widetilde{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then, it is easy to show, see Theorem 2.14, that the system satisfies (output energy preserving)

$$\frac{1}{2}\frac{d}{dt} \|x(t)\|^2_{\mathcal{L}} = -\langle \alpha y(t), y(t) \rangle_U = -\alpha_1 \, |(Tq)(b)|^2,$$

since $u(t) = 0$ for all $t \geq 0$, see (5.40). Note that the matrix α is not coercive, see (5.41), so we cannot use Theorem 5.17. However, we still can use Corollary 5.19. In this case, by using (5.40), we have

$$\|(\mathcal{L}x)(b)\|^2_{\mathbb{R}} = |(\rho^{-1}p)(b)|^2 + |(Tq)(b)|^2 = (\alpha_1^2 + 1)|(Tq)(b)|^2.$$

From this we clearly have (5.38) since $\langle \alpha\, y, y \rangle_U = \alpha_1 \, |(Tq)(b)|^2$. From this and Corollary 5.19 we can conclude that this system is exponentially stable (by the smoothness of $\mathcal{L}$, i.e., of $T(z)$ and $\rho(z)$). Since in this case, the system is output energy preserving we also have that the systems is exactly controllable and observable, see Theorem 3.24. $*$

Example 5.22 We consider again the Timoshenko beam, whose model together with the port-variables are described in Example 2.19 on page 42. Here we impose the boundary conditions used in Example 3.27 on page 81, i.e.,

$$\begin{aligned} &\frac{1}{\rho(a)}\, x_2(a,t) = 0, \quad \frac{1}{I_\rho(a)}\, x_4(a,t) = 0, \quad t \geq 0, \\ &K(b)\, x_1(b,t) = -\alpha_1 \frac{1}{\rho(b)}\, x_2(b,t), \quad EI(b)\, x_3(b,t) = -\alpha_2 \frac{1}{I_\rho(b)}\, x_4(b,t), \end{aligned} \tag{5.42}$$

where α_1 and α_2 are given positive gain feedback constants. Under this boundary conditions, i.e., $u(t) = 0$, the system satisfy the energy inequality (see Example 3.27 for details)

$$\frac{1}{2}\frac{d}{dt}\,\|x(t)\|^2_{\mathcal{L}} = -\langle \alpha y(t), y(t)\rangle_U = -\underbrace{\left(\alpha_1\,|(\rho^{-1}\,x_2)(b,t)|^2 + \alpha_2\,|(I_\rho^{-1}\,x_4)(b,t)|^2\right)}_{\langle \alpha y, y\rangle_U}. \tag{5.43}$$

(Recall that $U = \mathbb{R}^4$.) We prove that this system is exponentially stable, and we do this by using Corollary 5.19. Using the boundary conditions (5.42) we obtain

$$\begin{aligned} \|(\mathcal{L}x)(b)\|^2_{\mathbb{R}} &= |(kx_1)(b)|^2 + |(\rho^{-1}x_2)(b)|^2 + |(EIx_3)(b)|^2 + |(I_\rho^{-1}x_4)(b)|^2 \\ &= (\alpha_1^2+1)|(\rho^{-1}x_2)(b)|^2 + (\alpha_2^2+1)|(I_\rho^{-1}x_4)(b)|^2 \\ &\leq \kappa\,\langle \alpha y, y\rangle_{\mathbb{R}} \qquad \text{for some positive } \kappa, \end{aligned}$$

where we used (5.43). Provided that $K(z)$, $\rho(z)$, $EI(z)$, and $I_\rho(z)$ are continuously differentiable, we can conclude from Corollary 5.19 that the system is exponentially stable. Note that if either α_1 or α_2 is set to zero, we cannot conclude (from Corollary 5.19) anymore that the system is exponentially stable. $*$

Chapter 6

Systems with Dissipation

In Chapters 2 and 3 we have studied systems where the dissipation phenomena has been neglected. Recall that in Section 1.4 we mentioned that we would study a specific class of systems which might include dissipation, see equation (1.23). In this chapter we deal with a larger class of systems to those studied in Chapter 2 and we extend the results presented in that chapter to include this larger class. However, the results presented in Chapter 2 are fundamental, since the extended results depend on the results presented there. In brief, the class of systems studied in this chapter is

$$\frac{\partial x}{\partial t}(t,z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}x(t,z), \quad x(0,z) = x_0(z), \tag{6.1a}$$

$$u(t) = \mathcal{B}\mathcal{L}x(t,z), \quad z \in (a,b),\ t \geq 0 \tag{6.1b}$$

$$y(t) = \mathcal{C}\mathcal{L}x(t,z), \tag{6.1c}$$

where $\mathcal{B}$ and $\mathcal{C}$ are boundary operators and S and $\mathcal{L}$ are bounded coercive operators on $L_2(a,b;\mathbb{R}^m)$ and $X = L_2(a,b;\mathbb{R}^n)$, respectively. The differential operators $\mathcal{J}$ and $\mathcal{G}_R$ are given by

$$\mathcal{J}x = \sum_{i=0}^{N} P_i \frac{\partial^i x}{\partial z^i}, \quad \mathcal{G}_R x = \sum_{i=0}^{N} G_i \frac{\partial^i x}{\partial z^i}, \quad \mathcal{G}_R^* x = \sum_{i=0}^{N} (-1)^i G_i^T \frac{\partial^i x}{\partial z^i}, \tag{6.2}$$

with $\mathcal{G}_R^*$ being the formal adjoint operator of $\mathcal{G}_R$ and G_i, P_i, $i = \{1,2,\ldots,N\}$, are constant real matrices of size $n \times m$, and $n \times n$, respectively. Furthermore, it is assumed that these matrices satisfy

$$P_i = (-1)^{i+1} P_i^T, \quad i = 0,1,\ldots,N, \tag{6.3}$$

and either of the following conditions

$$\begin{bmatrix} P_N & G_N \\ G_N^T & 0 \end{bmatrix} \text{ has full rank,} \quad \text{if } G_i \neq 0 \text{ for at least one } i \in \{1, 2, \ldots, N\}; \tag{6.4a}$$

or

$$P_N \text{ has full rank,} \qquad \text{if } G_i = 0 \text{ for all } i \geq 1 \text{, i.e., if } \mathcal{G}_R = G_0 \,. \tag{6.4b}$$

It is worth to mention that the conditions above are not strong. In fact, many physical examples which occur in applications satisfy the conditions above. For instance, the heat equation satisfies condition (6.4a) and the wave equation with viscous damping satisfies (6.4b). Moreover, conditions (6.4a)–(6.4b) guarantee that we are imposing the appropriate number of boundary conditions. However, there are some cases in which the conditions above are not satisfied, e.g. when structural damping is used. Later we introduce some results which allow us to cover those cases as well.

Observe that Sturm-Liouville systems are a special class of this type of equations, choose $n = m = 1$. For more general n and m this class includes diffusion equations with control and observation through the boundary as well as flexible structures with damping. See Chapter 1 for more examples.

As we did in Chapter 2, in this chapter we also explain how to select the boundary operators $\mathcal{B}$ and $\mathcal{C}$ such that the system (6.1) is a boundary control system in the sense of Section 1.5. Furthermore, with this selection of $\mathcal{B}$ and $\mathcal{C}$ the system is dissipative as explained in Section 1.8. We also show that the selection of these boundary operators is based on the choice of a matrix, which in turn simplifies the analysis and design of this class of boundary control systems. Also, the relation with port-Hamiltonian systems (PHS) is studied, as well as the respective Dirac structure. We start describing the properties related to the skew-symmetric operator that describes the Dirac structure. These properties correspond to attributes coming from the internal interconnection of the elements that comprise the system. We also define the port-variables, which are the variables that the system uses to interact with the environment. In particular, we define the boundary port-variables. Finally, recall from Chapter 1 that a self-adjoint operator, L, is coercive if there exists an $\varepsilon > 0$ such that

$$\langle Lx, x\rangle = \langle x, Lx\rangle \geq \varepsilon \left\|x\right\|^2 > 0 \qquad \text{for all } x \in D(L), \tag{6.5}$$

i.e., L has a bounded inverse.

6.1. Relation with skew-symmetric operators

In this section we show that there is a skew-symmetric operator which is related to the class of systems described by (6.1). The idea of making this relation is that systems described by skew-symmetric operators have been studied extensively in the literature. In particular, we use the results presented in Chapter 2. Furthermore, this skew-symmetric operator describes the interconnection properties of the system.

The relation is as follows. Consider the operator given by

$$\mathcal{J}_e = \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix}, \qquad D(\mathcal{J}_e) = H^N(a,b,\mathbb{R}^n) \times H^N(a,b,\mathbb{R}^m), \tag{6.6}$$

where $\mathcal{J}$, $\mathcal{G}_R$, and $\mathcal{G}_R^*$ are given in (6.2). Note that if we define

$$\begin{bmatrix} f \\ f_r \end{bmatrix} = \mathcal{J}_e \begin{bmatrix} e \\ e_r \end{bmatrix} = \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{bmatrix} e \\ e_r \end{bmatrix}$$

and let $e_r = Sf_r$ with S an (coercive) operator on $L_2(a,b;\mathbb{R}^m)$, then we obtain

$$f = \mathcal{J}e - \mathcal{G}_R S \mathcal{G}_R^* e,$$

which is the same operator that appears in (6.1) with $e = \mathcal{L}x$. This idea has been used before to deal with dissipation in the framework of port-Hamiltonian systems, see for instance [vdSM02] and [MvdSM04]. Roughly speaking, $\mathcal{G}_R$ expresses how dissipation comes into the system, S describes the amount of dissipation in the system, and $\mathcal{J}_e$ gives the internal geometric properties of the system (6.1).

Basically, the operator $\mathcal{J}_e$ can be seen as the skew-symmetric operator describing the Dirac structure corresponding to system (6.1), see Theorem 2.7. In this case the Dirac structure will have an additional port (called the *resistive port*) with respect to the Dirac structure studied in Chapter 2, see Chapter 7 for more details. In order to define the dynamics of the (port-Hamiltonian) system we need to terminate this resistive port with a resistive relation as described above, see Figure 6.1.

The operator $\mathcal{J}_e$ plays a crucial role in the rest of this chapter. In the next section we show that the operator $\mathcal{J}_e$ is skew-symmetric. Furthermore, we also adapt the results appearing in Chapter 2 (or in [LZM05] and [LZM04]) to fit the operator $\mathcal{J}_e$ taking into account conditions (6.3), (6.4a), and (6.4b). That is, we adapt Theorem 2.1 to include conditions (6.3), (6.4a), and (6.4b).

6.2. Port-variables for skew-symmetric operators and boundary control systems related to $\mathcal{J}_e$

In this section we study the operator $\mathcal{J}_e$ introduced at the end of the previous section, i.e.,

$$\mathcal{J}_e = \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix}, \qquad D(\mathcal{J}_e) = H^N(a,b,\mathbb{R}^n) \times H^N(a,b,\mathbb{R}^m), \tag{6.7}$$

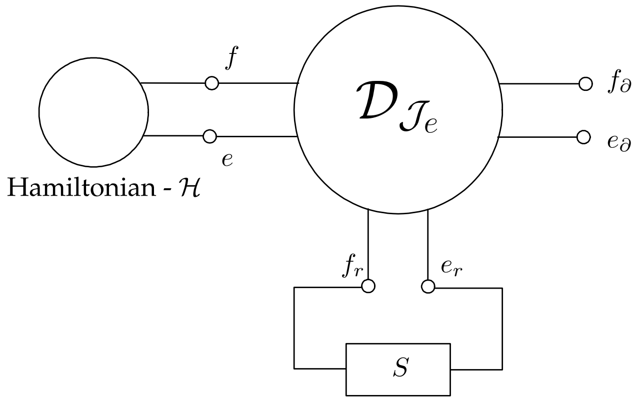

Figure 6.1.: Interconnection structure with resistive port.

where $\mathcal{J}$, $\mathcal{G}_R$, and $\mathcal{G}_R^*$ are given in (6.2). Furthermore these operators satisfy (6.3) and either (6.4a) or (6.4b). Note that assumption (6.3) imposed on the matrices P_i means that $\mathcal{J}$ is formally skew symmetric. Also, the validity of assumption (6.4a) means that $\mathcal{G}_R$ is a differential operator whose leading (matrix) coefficient, G_N, has full-rank. Condition (6.4b) implies that $\mathcal{G}_R$ is a bounded operator, i.e., $\mathcal{G}_R = G_0$, in which case we need the leading matrix P_N appearing in $\mathcal{J}$ to be nonsingular.

First, we prove that the operator $\mathcal{J}_e$ is formally skew-symmetric, which will allows us to use the results presented in Chapter 2.

Proposition 6.1: *The operator $\mathcal{J}_e$ defined by (6.7) together with (6.2) and (6.3) is formally skew-symmetric and can be written as:*

$$\mathcal{J}_e \begin{bmatrix} e \\ e_r \end{bmatrix} = \sum_{i=0}^{N} \underbrace{\begin{bmatrix} P_i & G_i \\ (-1)^{(i+1)} G_i^T & 0 \end{bmatrix}}_{\widetilde{P}_i} \frac{\partial^i}{\partial z^i} \begin{bmatrix} e \\ e_r \end{bmatrix} \tag{6.8}$$

where $\widetilde{P}_i \in \mathbb{R}^{(n+m)\times(n+m)}$ satisfies

$$\widetilde{P}_i = \begin{bmatrix} P_i & G_i \\ (-1)^{(i+1)} G_i^T & 0 \end{bmatrix} = (-1)^{i+1} \begin{bmatrix} P_i & G_i \\ (-1)^{i+1} G_i^T & 0 \end{bmatrix}^T = (-1)^{i+1} \widetilde{P}_i^T. \tag{6.9}$$

♡

PROOF: That $\mathcal{J}_e$ is formally skew-symmetric follows from the fact that $\mathcal{J}$ is formally skew-symmetric and $\mathcal{G}_R^*$ is the formal adjoint of $\mathcal{G}_R$. In fact, using this one obtains

$$\begin{aligned}
\langle \mathcal{J}_e x^1, x^2 \rangle &= \left\langle \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}, \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} \right\rangle \\
&= \langle \mathcal{J} x_1^1 + \mathcal{G}_R x_2^1, x_1^2 \rangle + \langle -\mathcal{G}_R^* x_1^1, x_2^2 \rangle \\
&= \langle x_1^1, -\mathcal{J} x_1^2 \rangle + \langle x_2^1, \mathcal{G}_R^* x_1^2 \rangle + \langle x_1^1, -\mathcal{G}_R x_2^2 \rangle \\
&= \left\langle \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}, - \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} \right\rangle \\
&= \langle x^1, -\mathcal{J}_e x^2 \rangle .
\end{aligned}$$

Using (6.2) into (6.7) we can see that $\mathcal{J}_e$ can be rewritten as

$$\begin{aligned}
\mathcal{J}_e \begin{bmatrix} e \\ e_r \end{bmatrix} &= \sum_{i=0}^{N} \begin{bmatrix} P_i & G_i \\ -(-1)^i G_i^T & 0 \end{bmatrix} \frac{\partial^i}{\partial z^i} \begin{bmatrix} e \\ e_r \end{bmatrix} \\
&= \sum_{i=0}^{N} \begin{bmatrix} P_i & G_i \\ (-1)^{(i+1)} G_i^T & 0 \end{bmatrix} \frac{\partial^i}{\partial z^i} \begin{bmatrix} e \\ e_r \end{bmatrix}.
\end{aligned}$$

Equation (6.9) follows easily from the definition of $\widetilde{P}_i$ and (6.3). ∎

Recall from Chapter 2 that we have parameterized the boundary conditions for which a formally skew-symmetric operator generates a contraction semigroup. We use those results to prove a similar result for the class of systems described by (6.1). First, we need to adapt conditions (6.4a) and (6.4b) imposed on the operator $\mathcal{J}_e$ to match the conditions that are used for the selection of the port-variables and the bilinear form (2.14). There, it was assumed that the leading coefficient matrix of the skew-symmetric operator, i.e., $\widetilde{P}_N$, was nonsingular, see Assumption 2.2 on page 25. This guarantees that the relation between the change of energy inside the spatial domain and the energy flowing through the boundary defines a nondegenerate bilinear form, see theorem below or Sections 2.1 and 2.2. This bilinear form is essential in the definition of the Dirac structure and of the boundary control systems which are obtained from this Dirac structure as it was seen in Chapter 2.

Since conditions (6.4a) and (6.4b) change the nonsingularity of $\widetilde{P}_N$, we need to redefine Theorem 2.1.

Theorem 6.2: *Let $\mathcal{J}_e$ be the skew symmetric operator defined by* (6.7) *together with* (6.2) *and (6.3). For any two functions $e_{e,i} = \left[\begin{smallmatrix} e_{1,i} \\ e_{p,i} \end{smallmatrix}\right] \in D(\mathcal{J}_e)$, $i \in \{1, 2\}$ we have:*

a. *If condition* (6.4a) *is satisfied then the following equation holds*

$$\int_a^b e_{e,1}^T(z)(\mathcal{J}_e e_{e,2})(z) + e_{e,2}^T(z)(\mathcal{J}_e e_{e,1})(z)dz = \tag{6.10}$$

$$\left[\begin{pmatrix} e_{e,1}^T(z), & \cdots & \frac{d^{N-1}e_{e,1}^T}{dz^{N-1}}(z) \end{pmatrix} Q \begin{pmatrix} e_{e,2}(z) \\ \vdots \\ \frac{d^{N-1}e_{e,2}}{dz^{N-1}}(z) \end{pmatrix}\right]_a^b,$$

where Q *is an* $(n+m)N \times (n+m)N$ *nonsingular symmetric matrix defined by*

$$Q = \begin{pmatrix} \widetilde{P}_1 & \widetilde{P}_2 & \widetilde{P}_3 & \cdots & \widetilde{P}_{N-1} & \widetilde{P}_N \\ -\widetilde{P}_2 & -\widetilde{P}_3 & -\widetilde{P}_4 & \cdots & -\widetilde{P}_N & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ (-1)^{N-1}\widetilde{P}_N & 0 & \cdots & \cdots & & 0 \end{pmatrix}, \tag{6.11}$$

with $\widetilde{P}_i$ *given in* (6.9).

b. *If condition* (6.4b) *is satisfied then the following equation holds*

$$\int_a^b e_{e,1}^T(z)(\mathcal{J}_e e_{e,2})(z) + e_{e,2}^T(z)(\mathcal{J}_e e_{e,1})(z)dz = \tag{6.12}$$

$$\left[\begin{pmatrix} e_{1,1}^T(z), & \cdots & \frac{d^{N-1}e_{1,1}^T}{dz^{N-1}}(z) \end{pmatrix} Q \begin{pmatrix} e_{1,2}(z) \\ \vdots \\ \frac{d^{N-1}e_{1,2}}{dz^{N-1}}(z) \end{pmatrix}\right]_a^b,$$

where Q *is an* $nN \times nN$ *nonsingular symmetric matrix defined by*

$$Q = \begin{pmatrix} P_1 & P_2 & P_3 & \cdots & P_{N-1} & P_N \\ -P_2 & -P_3 & -P_4 & \cdots & -P_N & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ (-1)^{N-1}P_N & 0 & \cdots & \cdots & & 0 \end{pmatrix}, \tag{6.13}$$

with P_i *given in* (6.2) *and* (6.3).

PROOF: In Theorem 2.1 a similar result is proved for any skew-symmetric operator with nonsingular leading coefficient matrix. This is proved by using integration by parts iteratively, see [LZM04, Theorem 3.1]. Here we use that result.

a. Assuming that condition (6.4a) is satisfied, we see that $\widetilde{P}_N$ is nonsingular.

Hence the conditions on Theorem 2.1 are satisfied, and from there the result follows.

b. We prove the result in the case $G_i = 0$ for all $i \geq 0$, or which is equivalent, when $\mathcal{G}_R = G_0$, see (6.4b). In that case, we have

$$\begin{aligned}
&\int_a^b e_{e,1}^T(z)(\mathcal{J}_e e_{e,2})(z) + e_{e,2}^T(z)(\mathcal{J}_e e_{e,1})(z)dz \\
&= \int_a^b \begin{bmatrix} e_{1,1} \\ e_{p,1} \end{bmatrix}^T \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{bmatrix} e_{1,2} \\ e_{p,2} \end{bmatrix} \\
&= + \begin{bmatrix} e_{1,2} \\ e_{p,2} \end{bmatrix}^T \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{bmatrix} e_{1,1} \\ e_{p,1} \end{bmatrix} dz.
\end{aligned}$$

After simplifying we obtain

$$\begin{aligned}
\int_a^b e_{e,1}^T(z)(\mathcal{J}_e e_{e,2})(z) + e_{e,2}^T(z)(\mathcal{J}_e e_{e,1})(z)dz = \int_a^b e_{1,1}^T(\mathcal{J}\, e_{1,2} + G_0 e_{p,2})\, dz \\
+ \int_a^b e_{p,1}^T(-G_0^T e_{1,2}) + e_{1,2}^T(\mathcal{J}\, e_{1,1} + G_0 e_{p,1}) + e_{p,2}^T(-G_0^T e_{1,1})\, dz.
\end{aligned}$$

Since $e_{1,1}^T G_0 e_{p,2} = e_{p,2}^T G_0^T e_{1,1}$ and $e_{1,2}^T G_0 e_{p,1} = e_{p,1}^T G_0^T e_{1,2}$, the equation above becomes

$$\begin{aligned}
&\int_a^b e_{e,1}^T(z)(\mathcal{J}_e e_{e,2})(z) + e_{e,2}^T(z)(\mathcal{J}_e e_{e,1})(z)dz \\
&\qquad = \int_a^b e_{1,1}^T(\mathcal{J}\, e_{1,2}) + e_{1,2}^T(\mathcal{J}\, e_{1,1})\, dz.
\end{aligned}$$

Hence, we have changed the relation on the extended skew-symmetric operator $\mathcal{J}_e$ to the skew-symmetric operator $\mathcal{J}$ with a nonsingular leading coefficient matrix P_N. Then, we can apply again Theorem 2.1 to $\mathcal{J}$ to obtain the result with the new Q matrix. ∎

Remark 6.3. Theorem 6.2 can be considered as an extension of Stokes Theorem to a class of skew-symmetric operators. Note that it induces a symmetric pairing on the boundary variables. Also, observe that the dimension of the symmetric matrix Q, and hence the dimension of the vector containing the boundary variables, changes according to condition (6.4a) or (6.4b). This is due to the fact that if $\mathcal{G}_R = G_0$, i.e., $\mathcal{G}_R$ is a bounded operator, it does not add boundary variables in the integration by parts (the part corresponding to $e_{p,i}$ does not appear on the right hand side of (6.12)). Obviously, if $\mathcal{G}_R$ is a differential operator, then the part corresponding to $e_{p,i}$ will appear in the boundary variables as can be seen on (6.10). It is worth to mention that once the operator $\mathcal{G}_R$ is chosen, then the matrix Q is fixed. ♣

Following the theorem above, it is easy to see that the results of Chapter 2 apply to the operator $\mathcal{J}_e$. That is why the remaining of this section is dedicated to recollect and adapt to $\mathcal{J}_e$ some of the contents that appear in that chapter. All those results are needed in the next section to prove the main ideas of this chapter.

6.2.1. Definition of boundary port-variables

Recall from Section 2.1 that in order to define the boundary port-variables it was necessary to introduce some new matrices, i.e., R_{ext} and Σ, see Lemma 2.4. It is clear that those matrices depend on Q (and its dimension). Following Theorem 6.2 it is easy to see that Lemma 2.4 still holds for the operator $\mathcal{J}_e$ once the matrix Q has been described. The only thing that changes is the dimension of R_{ext} and Σ accordingly to the dimension of the matrix Q, see Theorem 6.2. Therefore, the selection of the boundary port-variables follows easily once this dimension has been set.

Definition 6.4. The *boundary port-variables* associated with the differential operator $\mathcal{J}_e$ of Theorem 6.2 are the vectors e_∂, f_∂ defined as follows.

a. If condition (6.4a) is satisfied then

$$\begin{bmatrix} f_{\partial,e_e} \\ e_{\partial,e_e} \end{bmatrix} = R_{\text{ext}}\tau\left(\begin{bmatrix} e_1 \\ e_r \end{bmatrix}\right), \tag{6.14}$$

where $R_{\text{ext}} \in \mathbb{R}^{2(n+m)N\times 2(n+m)N}$ is defined according to Lemma 2.4 and $e_e = \left[\begin{smallmatrix} e_1 \\ e_r \end{smallmatrix}\right]$ is defined according to Theorem 6.2. Furthermore, $e_\partial, f_\partial \in \mathbb{R}^{(n+m)N}$ and the boundary trace operator $\tau : H^N(a,b;\mathbb{R}^{n+m}) \to \mathbb{R}^{2(n+m)N}$ is described in Definition 2.5.

b. If condition (6.4b) is satisfied then

$$\begin{bmatrix} f_{\partial,e_e} \\ e_{\partial,e_e} \end{bmatrix} = R_{\text{ext}}\tau(e_1) = \begin{bmatrix} f_{\partial,e_1} \\ e_{\partial,e_1} \end{bmatrix}, \tag{6.15}$$

where $R_{\text{ext}} \in \mathbb{R}^{2nN\times 2nN}$ is defined according to Lemma 2.4 and $e_e = \left[\begin{smallmatrix} e_1 \\ e_r \end{smallmatrix}\right]$ is defined according to Theorem 6.2. Furthermore, $e_\partial, f_\partial \in \mathbb{R}^{nN}$ and the boundary trace operator $\tau : H^N(a,b;\mathbb{R}^n) \to \mathbb{R}^{2nN}$ is described in Definition 2.5. ♣

Since we have defined the boundary ports corresponding to the skew-symmetric operator $\mathcal{J}_e$ it follows easily that the bilinear form used to define the Dirac structure is still described by (2.14) with $\mathcal{F} = \mathcal{E} = L_2(a,b;\mathbb{R}^{n+m})\times$ (dimension of f_∂), where the only difference is the dimension of the port-variables, which may change with respect to that of Section 2.2. However, we stress that this dimension is fixed once we define the operator $\mathcal{J}_e$ (and hence Q). Therefore, all the results in Sections 2.2 and 2.3 are valid for the operator $\mathcal{J}_e$. In particular, the Dirac structure is still described as in Theorem 2.7, which adapted to this case becomes

Theorem 6.5: *Consider the skew-symmetric operator $\mathcal{J}_e$ given by* (6.7) *together with* (6.2) *and (6.3). Let the boundary port-variables be described as in Definition* 6.4. *Then, the subspace $D_{\mathcal{J}_e}$ of $\mathcal{B}$ defined by*

$$D_{\mathcal{J}_e} = \left\{ \begin{bmatrix} f \\ f_\partial \\ e \\ e_\partial \end{bmatrix} \in \mathcal{B} \;\middle|\; e \in D(\mathcal{J}_e),\ \mathcal{J}e = f,\ \begin{bmatrix} f_{\partial,e} \\ e_{\partial,e} \end{bmatrix} = R_{\mathrm{ext}}\,\tau(e) \right\} \tag{6.16}$$

is a Dirac structure, where R_{ext} and $\tau(\cdot)$ are given according to Definition 6.4.

6.2.2. Definition of a class of boundary control systems related to $\mathcal{J}_e$

In the previous subsection we defined the boundary port-variables and the Dirac structure which are related to the extended operator $\mathcal{J}_e$. We also concluded that the results of Section 2.3 are also valid for $\mathcal{J}_e$ based on the modified Stokes Theorem (Theorem 6.2) and the redefined boundary port-variables, i.e., Definition 6.4. In particular, in this subsection we rewrite Theorem 2.14 adapted to the skew-symmetric operator $\mathcal{J}_e$ under assumption (6.4a) or (6.4b).

In other words, in this subsection we consider systems described by the following PDE

$$\frac{\partial x}{\partial t}(t) = \mathcal{J}_e \tilde{\mathcal{L}} x(t),$$

where $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{smallmatrix}\right]$ is a coercive operator on the state space defined by

$$\begin{aligned} &\tilde{X} = L_2(a,b)^n \times L_2(a,b)^m \text{ with inner product} \\ &\langle x_1, x_2 \rangle_{\tilde{\mathcal{L}}} = \langle u_1, \mathcal{L}_1 u_2 \rangle + \langle w_1, \mathcal{L}_2 w_2 \rangle \text{ for any } x_i = \left[\begin{smallmatrix} u_i \\ w_i \end{smallmatrix}\right] \in \tilde{X}, \text{ i=\{1,2\}}, \\ &\text{and corresponding norm } \|x_1\|^2_{\tilde{\mathcal{L}}} = \langle x_1, x_1 \rangle_{\tilde{\mathcal{L}}}, \end{aligned} \tag{6.17}$$

where $\langle \cdot, \cdot \rangle$ is the natural L_2-inner product. As previously mentioned, the idea is to define systems with inputs and outputs acting through the boundary of the spatial domain. The following theorem, which is an adaptation of Theorem 2.14, characterizes those inputs and outputs for which the system is energy preserving.

Theorem 6.6: *Let k be the dimension of f_∂ and e_∂ according to Definition 6.4, i.e., $k = (n+m)N$ or $k = nN$, and let W be a $k \times 2k$ matrix. If W has full rank and satisfies $W\Sigma W^T \geq 0$, where Σ is defined in (2.9), then the following system, with $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L}_1 & 0 \\ 0 & I \end{smallmatrix}\right]$,*

$$\frac{\partial x}{\partial t}(t) = \mathcal{J}_e \tilde{\mathcal{L}} x(t), \quad \text{or equivalently} \quad \left(\dot{x}(t),\, f_{\partial,\tilde{\mathcal{L}}x}(t),\, \tilde{\mathcal{L}}x(t),\, e_{\partial,\tilde{\mathcal{L}}x}(t)\right) \in \mathcal{D}_{\mathcal{J}_e}$$

defined on the state space $\tilde{X}$ (see (6.17)) with input

$$u(t) = W \begin{bmatrix} f_{\partial,\tilde{\mathcal{L}}x}(t) \\ e_{\partial,\tilde{\mathcal{L}}x}(t) \end{bmatrix}$$

is a boundary control system on $\tilde{X}$. Furthermore, the operator $\mathcal{A}_{\text{ext}} = \mathcal{J}_e \tilde{\mathcal{L}}$ with domain

$$D(\mathcal{A}_{\text{ext}}) = \left\{ \tilde{x} = \begin{bmatrix} x \\ x_r \end{bmatrix} \in \tilde{X} \,\middle|\, \tilde{\mathcal{L}}\tilde{x} \in \begin{bmatrix} H^N(a,b;\mathbb{R}^n) \\ H^N(a,b;\mathbb{R}^m) \end{bmatrix} \text{ and } \begin{bmatrix} f_{\partial,\tilde{\mathcal{L}}\tilde{x}} \\ e_{\partial,\tilde{\mathcal{L}}\tilde{x}} \end{bmatrix} \in \ker W \right\} \tag{6.18}$$

generates a contraction semigroup on $\tilde{X}$.

Let $\widetilde{W}$ be a full rank matrix of size $k \times 2k$ with $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ invertible. If we define the linear mapping $\mathcal{C} : \tilde{\mathcal{L}}^{-1}\big(H^N(a,b;\mathbb{R}^n) \times H^N(a,b;\mathbb{R}^m)\big) \to \mathbb{R}^k$ as,

$$\mathcal{C}x(t) := \widetilde{W} \begin{bmatrix} f_{\partial,\tilde{\mathcal{L}}x}(t) \\ e_{\partial,\tilde{\mathcal{L}}x}(t) \end{bmatrix} \tag{6.19}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{6.20}$$

then for $u \in C^2(0,\infty;\mathbb{R}^k)$, $\tilde{\mathcal{L}}x(0) \in H^N(a,b;\mathbb{R}^n) \times H^N(a,b;\mathbb{R}^m)$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\tilde{\mathcal{L}}} = \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{6.21}$$

where

$$P^{-1}_{W,\tilde{W}} = \begin{bmatrix} W\Sigma W^T & W\Sigma \widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{bmatrix}. \tag{6.22}$$

Furthermore, we have that the matrix $\left(\begin{smallmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{smallmatrix}\right)$ is invertible if and only if $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ is invertible.

Observe that the input and output of this class of distributed parameter systems is acting through the boundary of the spatial domain. Also this input and output are parameterized by the selection of a matrix. Furthermore, the rate of change of the energy of the system is a linear combination of the input and output, see (6.21). This makes it easy to define impedance passive and scattering passive systems. For more information on this class of systems refer to Chapter 2.

6.3. Port-variables and BCS for systems with dissipation

Now we turn to systems described by (6.1) where it is assumed that (6.3) holds together with either (6.4a) or (6.4b). Recall that the state space X is defined in (2.33) (note that it is different from $\tilde{X}$).

Before stating the main result, we stress that if we define

$$\begin{bmatrix} f \\ f_r \end{bmatrix} = \mathcal{J}_e \begin{bmatrix} e \\ e_r \end{bmatrix} = \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{bmatrix} e \\ e_r \end{bmatrix}$$

and let $e_r = Sf_r = -S\mathcal{G}_R^* e$ with S a coercive operator, we obtain

$$f = \mathcal{J}e - \mathcal{G}_R S \mathcal{G}_R^* e,$$

which is the same operator that defines our class of systems, see Section 6.1. As mentioned earlier, this idea of feedback will be used to prove the main results of this paper. First we introduce again the notion of boundary port-variables adapted to the class of systems described by (6.1). Then we proceed to prove the main results.

6.3.1. Boundary port-variables

In Subsection 6.2.1 we introduced the concept of boundary port-variables for a class of systems governed by the operator $\mathcal{J}_e$. In Section 6.1 we saw that this class of systems is related to systems described by (6.1) via the feedback transformation

$$e_r = Sf_r = -S\mathcal{G}_R^* e, \qquad \text{where } e = \mathcal{L}x. \tag{6.23}$$

Using the equation above, then it is easy to see from Definition 6.4 that the port-variables associated with systems given by (6.1) can be defined as follows.

Definition 6.7. The *boundary port-variables* associated with the differential operator $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$ given by (6.2)–(6.3) are the vectors $g_{e\partial}$, $g_{f\partial}$ defined as follows

a. If condition (6.4a) is satisfied then

$$\begin{bmatrix} g_{f\partial,e} \\ g_{e\partial,e} \end{bmatrix} = R_{\text{ext}}\,\tau\left(\begin{bmatrix} e \\ -S\mathcal{G}_R^* e \end{bmatrix}\right), \tag{6.24}$$

where $R_{\text{ext}} \in \mathbb{R}^{2(n+m)N\times 2(n+m)N}$ is defined according to Lemma 2.4 and the boundary trace operator $\tau : H^N(a,b;\mathbb{R}^{n+m}) \to \mathbb{R}^{2(n+m)N}$ is described in Definition 2.5. Furthermore, $g_{e\partial}, g_{f\partial} \in \mathbb{R}^{(n+m)N}$.

b. If condition (6.4b) is satisfied, then

$$\begin{bmatrix} g_{f\partial,e} \\ g_{e\partial,e} \end{bmatrix} = R_{\text{ext}}\,\tau(e), \tag{6.25}$$

where $R_{\text{ext}} \in \mathbb{R}^{2nN\times 2nN}$ is defined according to Lemma 2.4. Furthermore, $g_{e\partial}, g_{f\partial} \in \mathbb{R}^{nN}$ and the boundary trace operator $\tau : H^N(a,b;\mathbb{R}^n) \to \mathbb{R}^{2nN}$ is described in Definition 2.5. ♣

Remark 6.8. Observe that the definition above is the same as Definition 6.4 whenever $e_r = -S\mathcal{G}_R^* e$. Thus, these boundary port-variables play a similar roll to those in Section 6.2 and Chapter 2. ♣

6.3.2. Boundary control systems related to $(\mathcal{J} - \mathcal{G}_R S\mathcal{G}_R^*)\mathcal{L}$

Next we proceed to define boundary control systems for the class given by (6.1). First we prove that if we set the input $u(t)$ to zero in (6.1) then the resulting PDE with boundary conditions $\mathcal{BL}x = 0$ will have a unique (classical or weak) solution. That result is given by the following theorem.

Theorem 6.9: *Let k be the dimension of $g_{f\partial}$ and $g_{e\partial}$ according to Definition 6.7, i.e., $k = (n+m)N$ or $k = nN$, let W be a $k \times 2k$ matrix, and let X be given by* (2.33). *Consider the operator $\mathcal{A}_g = (\mathcal{J} - \mathcal{G}_R S\mathcal{G}_R^*)\mathcal{L}$ with domain*

$$D(\mathcal{A}_g) = \left\{ \mathcal{L}x \in H^N(a,b;\mathbb{R}^n) \,\middle|\, S\mathcal{G}_R^*\mathcal{L}x \in H^N(a,b;\mathbb{R}^m),\ \begin{bmatrix} g_{f\partial,\mathcal{L}x} \\ g_{e\partial,\mathcal{L}x} \end{bmatrix} \in \ker W \right\}. \tag{6.26}$$

If W has full rank and satisfies $W\Sigma W^T \geq 0$, where Σ is given by (2.9)*, then $\mathcal{A}_g$ generates a contraction semigroup on X and it satisfies*

$$\langle \mathcal{A}_g\, x, x\rangle_{\mathcal{L}} \leq -\left\langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\right\rangle, \quad \forall\, x \in D(\mathcal{A}_g). \tag{6.27}$$

PROOF: As mentioned before, the proof is based on a feedback argument on the operator $\mathcal{J}_e$. We make use of its corresponding semigroup generator $\mathcal{A}_{\text{ext}}$ described in Theorem 6.6. Since $\mathcal{A}_{\text{ext}}$ is the generator of a contraction semigroup (see Theorem 6.6) on $\tilde{X}$ (see (6.17)) we have from the Lümer-Phillips theorem (see [Paz83, §1.4 and §3.3]) that

$$\langle \mathcal{A}_{\text{ext}}\tilde{x}, \tilde{x}\rangle_{\tilde{\mathcal{L}}} \leq 0 \quad \text{for all } \tilde{x} \in D(\mathcal{A}_{\text{ext}}) \text{ and} \tag{6.28}$$

$$\text{ran}\,(\lambda I - \mathcal{A}_{\text{ext}}) = \tilde{X} \quad \text{for some } \lambda > 0. \tag{6.29}$$

Now we can proceed to prove that $\mathcal{A}_g$ generates a contraction semigroup on X. To do so, we use the same Lümer-Phillips theorem. That is, we first prove that $\mathcal{A}_g$ satisfies $\langle \mathcal{A}_g\, x, x\rangle_{\mathcal{L}} \leq 0$ for any $x \in D(\mathcal{A}_g)$ and next that $\text{ran}\,(\lambda I - \mathcal{A}_g) = X$ for some $\lambda > 0$. For $x \in D(\mathcal{A}_g)$, we have

$$\langle \mathcal{A}_g\, x, x\rangle_{\mathcal{L}} = \langle (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}x, \mathcal{L}x\rangle = \langle \mathcal{J}\mathcal{L}x, \mathcal{L}x\rangle + \langle -\mathcal{G}_R S \mathcal{G}_R^* \mathcal{L}x, \mathcal{L}x\rangle\,.$$

Define $x_r = -S\mathcal{G}_R^*\mathcal{L}x$ and observe that $x_r \in H^N(a, b; \mathbb{R}^m)$, see (6.26). It is now easy to see that $\left[\begin{smallmatrix} x \\ x_r \end{smallmatrix}\right] \in D(\mathcal{A}_{\text{ext}})$ since $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L} & 0 \\ 0 & I \end{smallmatrix}\right]$, see Remark 6.8 and (6.18). From this, the equation above, and since $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L} & 0 \\ 0 & I \end{smallmatrix}\right]$ we can see that

$$\begin{aligned}
\langle \mathcal{A}_g\, x, x\rangle_{\mathcal{L}} &= \langle \mathcal{J}\mathcal{L}x + \mathcal{G}_R x_r, \mathcal{L}x\rangle \\
&= \langle \mathcal{J}\mathcal{L}x + \mathcal{G}_R x_r, \mathcal{L}x\rangle + \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \\
&= \langle \mathcal{J}\mathcal{L}x + \mathcal{G}_R x_r, \mathcal{L}x\rangle + \langle \mathcal{G}_R^*\mathcal{L}x, -x_r\rangle - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \\
&= \left\langle \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix}, \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix} \right\rangle - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \\
&= \left\langle \mathcal{A}_{\text{ext}} \begin{bmatrix} x \\ x_r \end{bmatrix}, \begin{bmatrix} x \\ x_r \end{bmatrix} \right\rangle_{\tilde{\mathcal{L}}} - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \\
&\leq - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \leq 0,
\end{aligned} \tag{6.30}$$

where in the third step we used $x_r = -S\mathcal{G}_R^*\mathcal{L}x$ and in the last step we used (6.28) and the fact that $S \geq 0$, see (6.5).

Next we prove the range condition on $\mathcal{A}_g$. That is, for a $\lambda > 0$ we have to show that for any given $f \in X$ we can find an $x \in D(\mathcal{A}_g)$ such that

$$f = (\lambda I - \mathcal{A}_g)x.$$

In order to prove this, let

$$P = \begin{bmatrix} 0 & 0 \\ 0 & S^{-1} - \lambda I \end{bmatrix}.$$

Since S is coercive, we can find some $\lambda > 0$ such that $S^{-1} - \lambda I \geq 0$. Thus we can assume that P is a nonnegative operator. It thus follows from Corollary 3.3

of [Paz83] that $\mathcal{A}_{\text{ext}} - P$ generates a contraction semigroup. This in turn implies (by the Lümer-Phillips theorem) that ran $(\lambda I - \mathcal{A}_{\text{ext}} + P) = \tilde{X}$. Thus, given any $\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] \in \tilde{X}$ we can find $\left[\begin{smallmatrix} x \\ x_r \end{smallmatrix}\right] \in D(\mathcal{A}_{\text{ext}})$ such that

$$\begin{aligned} \begin{bmatrix} f \\ 0 \end{bmatrix} &= (\lambda I - \mathcal{A}_{\text{ext}} + P) \begin{bmatrix} x \\ x_r \end{bmatrix} = \begin{bmatrix} \lambda \mathcal{L}^{-1} - \mathcal{J} & -\mathcal{G}_R \\ \mathcal{G}_R^* & S^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix} \\ \Rightarrow \quad & f = (\lambda \mathcal{L}^{-1} - \mathcal{J})\mathcal{L}x - \mathcal{G}_R x_r \quad \text{and} \\ & x_r = -S\mathcal{G}_R^* \mathcal{L}x \\ \Rightarrow \quad & f = [\lambda I - (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}]x. \end{aligned} \tag{6.31}$$

Since $\left[\begin{smallmatrix} x \\ x_r \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ -S\mathcal{G}_R^*\mathcal{L}x \end{smallmatrix}\right] \in D(\mathcal{A}_{\text{ext}})$, it is easy to see that $x \in D(\mathcal{A}_g)$. Then, from (6.31) we can see that $\mathcal{A}_g$ satisfies the range condition. Concluding, we see that $\mathcal{A}_g$ generates a contraction semigroup. ■

Remark 6.10. Note that the left-hand side of equation (6.27) involves the inner product on X, see (2.33), whereas the right-hand side involves the L_2-inner product. ♣

Observe that the coercivity of the operator S plays an important role in the proof of existence of solutions. As it will be seen later, this assumption is also important in the definition of BCS.

Following Theorem 6.6 and Theorem 6.9 we can prove the following result.

Theorem 6.11: *Let k be the dimension of $g_{f,\partial}$ and $g_{e,\partial}$ according to Definition 6.7, i.e., $k = (n+m)N$ or $k = nN$, and let W be a $k \times 2k$ matrix. If W has full rank and satisfies $W \Sigma W^T \geq 0$, where Σ is defined in (2.9), then the system*

$$\frac{\partial x}{\partial t}(t) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}x(t) \tag{6.32}$$

with input

$$u(t) = \mathcal{B}x(t) = W \begin{bmatrix} g_{f\partial,\mathcal{L}x}(t) \\ g_{e\partial,\mathcal{L}x}(t) \end{bmatrix} \tag{6.33}$$

is a boundary control system on X. Furthermore, the operator $\mathcal{A}_g = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}$ with domain

$$D(\mathcal{A}_g) = \left\{ \mathcal{L}x \in H^N(a,b;\mathbb{R}^n) \,\middle|\, S\mathcal{G}_R^*\mathcal{L}x \in H^N(a,b;\mathbb{R}^m),\ \begin{bmatrix} g_{f\partial,\mathcal{L}x} \\ g_{e\partial,\mathcal{L}x} \end{bmatrix} \in \ker W \right\}. \tag{6.34}$$

generates a contraction semigroup on X.

Let $\widetilde{W}$ be a full rank matrix of size $k \times 2k$ with $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ invertible. If we define the linear mapping $\mathcal{C} : \mathcal{L}^{-1} H^N(a, b; \mathbb{R}^n) \to \mathbb{R}^k$ as,

$$\mathcal{C}x(t) := \widetilde{W} \begin{bmatrix} g_{f\partial,\mathcal{L}x}(t) \\ g_{e\partial,\mathcal{L}x}(t) \end{bmatrix} \tag{6.35}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{6.36}$$

then for $u \in C^2(0, \infty; \mathbb{R}^k)$, $\mathcal{L}x(0) \in H^N(a, b, \mathbb{R}^n)$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} &= \frac{1}{2} \begin{bmatrix} u^T(t) & y^T(t) \end{bmatrix} P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} - \langle \mathcal{G}_R^* \mathcal{L}x(t), S\mathcal{G}_R^* \mathcal{L}x(t) \rangle \\ &\le \frac{1}{2} \begin{bmatrix} u^T(t) & y^T(t) \end{bmatrix} P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \end{aligned} \tag{6.37}$$

where

$$P^{-1}_{W,\tilde{W}} = \begin{bmatrix} W\Sigma W^T & W\Sigma \widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{bmatrix}. \tag{6.38}$$

Furthermore, we have that the matrix $\left(\begin{smallmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{smallmatrix}\right)$ is invertible if and only if $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ is invertible.

PROOF: We divide the proof in three steps. In Step 1. and 2. we show that we have a boundary control system in the sense of Section 1.5. In step 3. we prove (6.37) and (6.38), respectively. For a boundary control system we have to show that for zero inputs, the operator $\mathcal{A}_g$ generates a C_0-semigroup, and furthermore that there exists a bounded operator $\mathfrak{R}$ mapping into the domain of $\mathcal{B}$ and such that $\mathcal{B}\mathfrak{R}u = u$ for all $u \in \mathbb{R}^k$.

Step 1: As mentioned above, we have to show that $\mathcal{A}_g = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}$ with domain (6.34) is an infinitesimal generator of a semigroup on X. This follows directly from Theorem 6.9.

Step 2: We have to find a bounded linear operator $\mathfrak{R}$ such that $\mathcal{L}\mathfrak{R}u \in H^N(a, b)^n$ and $\mathcal{B}\mathfrak{R}u = u$ for all $u \in \mathbb{R}^k$. Since $\mathcal{B}(\cdot) = W \left[\begin{smallmatrix} g_{f\partial,\mathcal{L}\cdot} \\ g_{e\partial,\mathcal{L}\cdot} \end{smallmatrix}\right]$, the result follows from the coercitivity of $\mathcal{L}$ and the surjectivity of the boundary operator (see Theorem 2.6) and W. This also follows similarly as the second step in the proof of Theorem 4.5 of [LZM04].

Step 3: By the definition of $\mathfrak{R}$ and $D(\mathcal{A}_g)$, we see that the conditions stated in the theorem are the same as $x(0) - \mathfrak{R}u(0) \in D(\mathcal{A}_g)$. Hence by Theorem 3.3.3 of [CZ95b] we have that there exists a classical solution of (6.32)–(6.33). Hence, in particular, $\mathcal{L}x(t) \in H^N(a,b;\mathbb{R}^n)$ holds pointwise in t, $x(t)$ is differentiable as a function of t, and $\dot{x}(t) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}x(t)$. Using this, we obtain

$$\begin{aligned}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 =& \frac{d}{dt}\langle x(t), x(t)\rangle_{\mathcal{L}} \\ =& \langle \dot{x}(t), x(t)\rangle_{\mathcal{L}} + \langle x(t), \dot{x}(t)\rangle_{\mathcal{L}} \\ =& \langle (\mathcal{J} - \mathcal{G}_R S\mathcal{G}_R^*)\mathcal{L}x(t), \mathcal{L}x(t)\rangle + \langle \mathcal{L}x(t), (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}x(t)\rangle .\end{aligned}$$

Define $x_r = -S\mathcal{G}_R^*\mathcal{L}x$ and observe that $x_r \in H^N(a,b;\mathbb{R}^m)$, see (6.34). It is now easy to see that $\left[\begin{smallmatrix} x \\ x_r \end{smallmatrix}\right] \in D(\mathcal{A}_{\text{ext}})$ with $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L} & 0 \\ 0 & I \end{smallmatrix}\right]$, see Remark 6.8. From this, the equation above, and defining $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L} & 0 \\ 0 & I \end{smallmatrix}\right]$ we can see by using the coercivity of S, see (6.5), that

$$\begin{aligned}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 =& \langle \mathcal{J}\mathcal{L}x(t) + \mathcal{G}_R x_r(t), \mathcal{L}x(t)\rangle + \langle \mathcal{L}x(t), \mathcal{J}\mathcal{L}x(t) + \mathcal{G}_R x_r(t)\rangle \\ =& \langle \mathcal{J}\mathcal{L}x(t) + \mathcal{G}_R x_r(t), \mathcal{L}x(t)\rangle + \langle \mathcal{L}x(t), \mathcal{J}\mathcal{L}x(t) + \mathcal{G}_R x_r(t)\rangle \\ &+ \langle \mathcal{G}_R^*\mathcal{L}x(t), S\mathcal{G}_R^*\mathcal{L}x(t)\rangle + \langle S\mathcal{G}_R^*\mathcal{L}x(t), \mathcal{G}_R^*\mathcal{L}x(t)\rangle \\ &- 2\langle \mathcal{G}_R^*\mathcal{L}x(t), S\mathcal{G}_R^*\mathcal{L}x(t)\rangle .\end{aligned}$$

And using again the definition of $x_r = -S\mathcal{G}_R^*\mathcal{L}x$ we obtain

$$\begin{aligned}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 =& \langle \mathcal{J}\mathcal{L}x(t) + \mathcal{G}_R x_r(t), \mathcal{L}x(t)\rangle + \langle \mathcal{L}x(t), \mathcal{J}\mathcal{L}x(t) + \mathcal{G}_R x_r(t)\rangle \\ &+ \langle \mathcal{G}_R^*\mathcal{L}x(t), -x_r(t)\rangle + \langle -x_r(t), \mathcal{G}_R^*\mathcal{L}x(t)\rangle - 2\langle \mathcal{G}_R^*\mathcal{L}x(t), S\mathcal{G}_R^*\mathcal{L}x(t)\rangle \\ =& \left\langle \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix}\begin{bmatrix} \mathcal{L}x(t) \\ x_r(t) \end{bmatrix}, \begin{bmatrix} \mathcal{L}x(t) \\ x_r(t) \end{bmatrix}\right\rangle \\ &+ \left\langle \begin{bmatrix} \mathcal{L}x(t) \\ x_r(t) \end{bmatrix}, \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix}\begin{bmatrix} \mathcal{L}x(t) \\ x_r(t) \end{bmatrix}\right\rangle - 2\langle \mathcal{G}_R^*\mathcal{L}x(t), S\mathcal{G}_R^*\mathcal{L}x(t)\rangle \\ =& \left\langle \frac{d}{dt}\begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}, \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}\right\rangle_{\tilde{\mathcal{L}}} + \left\langle \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}, \frac{d}{dt}\begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}\right\rangle_{\tilde{\mathcal{L}}} \\ &- 2\langle \mathcal{G}_R^*\mathcal{L}x(t), S\mathcal{G}_R^*\mathcal{L}x(t)\rangle \\ =& \frac{d}{dt}\left\|\begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}\right\|_{\tilde{\mathcal{L}}}^2 - 2\langle \mathcal{G}_R^*\mathcal{L}x(t), S\mathcal{G}_R^*\mathcal{L}x(t)\rangle ,\end{aligned}$$

where we used the fact that $\frac{d}{dt}\left[\begin{smallmatrix} x(t) \\ x_r(t) \end{smallmatrix}\right] = \mathcal{J}_e\tilde{\mathcal{L}}\left[\begin{smallmatrix} x(t) \\ x_r(t) \end{smallmatrix}\right]$ (see Theorem 6.6). The rest of the proof follows from equations (6.21) and (6.22) and the fact that $\left[\begin{smallmatrix} x \\ x_r \end{smallmatrix}\right] \in D(\mathcal{A}_{\text{ext}})$. ■

Note that we now have similar results to those of Chapter 2. The main difference being that these systems satisfy an energy inequality, unlike Section 2.3 where the systems were energy preserving. In the next section we discuss some examples, which should help to illustrate the main ideas presented here.

6.4. Some examples

Example 6.12 (Fixed bed reactor) Consider the fixed bed reactor of Example 1.5. The main phenomena which takes place into the reactor are the diffusion and the convection. The resulting PDE is

$$\frac{\partial C}{\partial t}(t,z) = D\frac{\partial^2 C}{\partial z^2}(t,z) - U\frac{\partial C}{\partial z}(t,z), \tag{6.39}$$

where $U > 0$ is the velocity of the fluid and $D > 0$ the diffusion constant. Note that the heat equation is obtained if we let $U = 0$, see Example 1.4. Comparing the equation above with (6.1), we can easily see that in this case we have

$$\mathcal{J} = -U\frac{\partial}{\partial z}, \qquad \mathcal{L} = I, \qquad -\mathcal{G}_R S \mathcal{G}_R^* = D\frac{\partial^2}{\partial z^2}.$$

From this we get

$$\mathcal{G}_R = \frac{\partial}{\partial z}, \quad S = D, \quad \text{and} \quad \mathcal{G}_R^* = -\frac{\partial}{\partial z}, \tag{6.40}$$

and thus (see equations (6.2) and (6.9)) $N = 1$,

$$P_1 = -U,\ G_1 = 1,\ P_0 = G_0 = 0,\ \text{and}\ \widetilde{P}_1 = \begin{bmatrix} -U & 1 \\ 1 & 0 \end{bmatrix} = Q.$$

Recall that $\mathcal{G}_R^*$ is the formal adjoint of $\mathcal{G}_R$, i.e., the adjoint of $\mathcal{G}_R$ ignoring boundary variables. Note that in this case equation (6.4a) is satisfied. Then it is easy to see that equation (6.39) becomes

$$\frac{\partial C}{\partial t}(t,z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)C(t,z). \tag{6.41}$$

From Definition 6.7 and Lemma 2.4 we obtain the boundary port-variables

$$\begin{aligned}\begin{bmatrix} g_{f,\partial} \\ g_{e,\partial} \end{bmatrix} &= \frac{1}{\sqrt{2}}\begin{pmatrix} -U & 1 & U & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{bmatrix} C(b) \\ D\frac{\partial C}{\partial z}(b) \\ C(a) \\ D\frac{\partial C}{\partial z}(a) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} -U(C(b)-C(a)) + D(\frac{\partial C}{\partial z}(b) - \frac{\partial C}{\partial z}(a)) \\ C(b)-C(a) \\ C(b)+C(a) \\ D(\frac{\partial C}{\partial z}(b) + \frac{\partial C}{\partial z}(a)) \end{bmatrix}.\end{aligned} \tag{6.42}$$

Typically, the boundary conditions are chosen as a linear combination of UC and $D\frac{\partial C}{\partial z}$ known as Danckwerts boundary conditions, see [FA62]. In [SMJ$^+$99] the authors set

$$D\frac{\partial C}{\partial z}(t,a) - UC(t,a) = UC_{\text{in}}(t), \quad \text{and} \quad D\frac{\partial C}{\partial z}(t,b) = 0, \tag{6.43}$$

where C_{in} is an input function. It is easy to see that these boundary conditions can be obtained from the port-variables by premultiplying them by the following matrix

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & -U & 1 \\ 1 & U & 0 & 1 \end{pmatrix}.$$

Since this matrix satisfies $W \Sigma W^T = \left[\begin{smallmatrix} U & 0 \\ 0 & U \end{smallmatrix}\right] > 0$, we have that the results in Theorem 6.11 apply to this system. Note that the domain of the semigroup generator in this case is, see (6.26),

$$\begin{aligned} D(\mathcal{A}_g) &= \left\{ x \in H^1(a,b) \,\middle|\, D\frac{\partial x}{\partial z} \in H^1(a,b), \begin{bmatrix} g_{f\partial,\mathcal{L}x} \\ g_{e\partial,\mathcal{L}x} \end{bmatrix} \in \ker W \right\} \\ &= \left\{ x \in H^2(a,b) \,\middle|\, D\frac{\partial x}{\partial z}(a) - Ux(a) = 0, D\frac{\partial x}{\partial z}(b) = 0 \right\}. \end{aligned}$$

In particular, assume we selected the output

$$y(t) = \frac{\sqrt{2}}{2} \begin{bmatrix} -D\frac{\partial C}{\partial z}(t,a) \\ D\frac{\partial C}{\partial z}(t,b) - UC(t,b) \end{bmatrix}. \tag{6.44}$$

The corresponding $\widetilde{W}$ is

$$\widetilde{W} = \frac{1}{2} \begin{pmatrix} 1 & U & 0 & -1 \\ 1 & 0 & -U & 1 \end{pmatrix}.$$

Following equations (6.37) and (6.38) we obtain that the energy of this system, $E(t) = \frac{1}{2}\|x(t)\|^2$, satisfies

$$\frac{d}{dt}E(t) = \frac{1}{2U}\|u(t)\|^2_{\mathbb{R}} - \frac{1}{U}\|y(t)\|^2_{\mathbb{R}} - D\left\|\frac{\partial x}{\partial z}\right\|^2_X, \tag{6.45}$$

where the last term corresponds to $\langle \mathcal{G}_R^* \mathcal{L}x, S\mathcal{G}_R^* \mathcal{L}x \rangle$. $*$

Example 6.13 (Timoshenko beam with damping) We study a linear system of Timoshenko type beam equations with frictional dissipative terms. This system consist of a model for vibrating beams subjected to two frictional mechanisms. The transverse vibrations of a beam are given by two coupled partial differential equations (compare with Examples 1.2 and 2.19)

$$\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial z}\left[K\left(\frac{\partial w}{\partial z} - \phi\right)\right] - \frac{\partial w}{\partial t}, \quad z \in (a,b),\ t \geq 0, \tag{6.46}$$

$$I_\rho \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial z}\left(EI\frac{\partial \phi}{\partial z}\right) + K\left(\frac{\partial w}{\partial z} - \phi\right) - \frac{\partial \phi}{\partial t}. \tag{6.47}$$

The function $w(t,z)$ is the transverse displacement of the beam and $\phi(t,z)$ is the rotation angle of a filament of the beam. The coefficients $\rho(z)$, $I_\rho(z)$, $E(z)$, $I(z)$, and $K(z)$ are the mass per unit length, the polar moment of inertia of a cross section, Youngs modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively. See [RFSC05] and the references therein for more details on this model.

As state variables we choose

$$\begin{aligned} x_1 &= \tfrac{\partial w}{\partial z} - \phi : && \text{shear displacement,} \\ x_2 &= \rho \tfrac{\partial w}{\partial t} : && \text{transverse momentum distribution,} \\ x_3 &= \tfrac{\partial \phi}{\partial z} : && \text{angular displacement,} \\ x_4 &= I_\rho \tfrac{\partial \phi}{\partial t} : && \text{angular momentum distribution.} \end{aligned}$$

Then the model of the beam can be rewritten as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{\partial}{\partial z} & 0 & -1 \\ \frac{\partial}{\partial z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial z} \\ 1 & 0 & \frac{\partial}{\partial z} & 0 \end{bmatrix}}_{\mathcal{J}} \begin{bmatrix} K\, x_1 \\ \frac{1}{\rho}\, x_2 \\ EI\, x_3 \\ \frac{1}{I_\rho}\, x_4 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathcal{G}_R S \mathcal{G}_R^*} \begin{bmatrix} K\, x_1 \\ \frac{1}{\rho}\, x_2 \\ EI\, x_3 \\ \frac{1}{I_\rho}\, x_4 \end{bmatrix}. \tag{6.48}$$

From here we see $\mathcal{J}$ and $\mathcal{G}_R S \mathcal{G}_R^*$, and from this it follows that $N = 1$, $n = 4$, $m = 2$,

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{G}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = G_0,$$

$\mathcal{G}_R^* = G_0^T$, $S = I$, and $\mathcal{L} = \operatorname{diag}\{K, \frac{1}{\rho}, EI, \frac{1}{I_\rho}\}$. In this case $\mathcal{G}_R = G_0$ and clearly this system satisfies condition (6.4b), i.e., P_1 is nonsingular. In [RFSC05] the beam is assumed to be clamped at both sides. This corresponds to the following boundary conditions (inputs)

$$\frac{1}{\rho(a)}\, x_2(a) = \frac{1}{I_\rho(a)}\, x_4(a) = \frac{1}{\rho(b)}\, x_2(b) = \frac{1}{I_\rho(b)}\, x_4(b) = 0. \tag{6.49}$$

The boundary port-variables are given by (2.55). From this we can see that W can be selected as

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad W \Sigma W^T = 0.$$

As output we can choose

$$y = \begin{bmatrix} -K(a)x_1(a) \\ -(EI)(a)x_3(a) \\ K(b)x_1(b) \\ (EI)(b)x_3(b) \end{bmatrix}, \quad \text{with} \quad \widetilde{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Following equations (6.37) and (6.38) we obtain that the energy of this system, $E(t) = \frac{1}{2}\|x(t)\|^2_{\mathcal{L}}$, satisfies

$$\frac{d}{dt}E(t) \le u^T(t)\, y(t).$$

*

6.5. A larger class of systems

In Chapter 2 it was assumed that the matrix Q appearing in Theorem 2.1 was nonsingular, see Assumption 2.2. Similarly, conditions (6.4a)–(6.4b) were imposed in order to guarantee that the resulting bilinear form in Theorem 6.2 is nondegenerate, i.e., that the resulting Q is nonsingular. However, there are some cases in which these conditions are not satisfied. The following example shows one of those situations.

Example 6.14 Consider a vibrating string with structural damping. Structural damping is the use of internal friction in a material to change vibrational energy into heat. This reduces excessive noise and vibration by converting them into heat to be expelled into the surrounding area. In this case the model becomes (see Example 1.1)

$$\frac{\partial^2 u}{\partial t^2}(z,t) = c\frac{\partial^2 u}{\partial z^2}(z,t) + \frac{\partial^3 u}{\partial z^2 \partial t}(z,t), \quad c = \frac{T}{\rho}.$$

This can be written as an evolution equation as we did in Example 1.6, see (1.15), as follows

$$\frac{\partial}{\partial t}\begin{bmatrix} p \\ q \end{bmatrix}(z,t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial z}\begin{bmatrix} \frac{1}{\rho}p \\ Tq \end{bmatrix}(z,t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{\partial^2}{\partial z^2}\begin{bmatrix} \frac{1}{\rho}p \\ Tq \end{bmatrix}(z,t). \quad (6.50)$$

Since this is a system with dissipation we need to consider the operator describing the interconnection structure, i.e., $\mathcal{J}_e$. In this case we have that $\mathcal{J} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\frac{\partial}{\partial z}$, $S = 1$, $\mathcal{G}_R = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\frac{\partial}{\partial z}$, and $\mathcal{G}_R^* = -\begin{bmatrix} 1 & 0 \end{bmatrix}\frac{\partial}{\partial z}$. This implies that the extended operator $\mathcal{J}_e = \left[\begin{smallmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{smallmatrix}\right]$ is described by (see Proposition 6.1)

$$\begin{bmatrix} f_1 \\ f_2 \\ f_r \end{bmatrix} = \underbrace{\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right)}_{\widetilde{P}_1}\frac{\partial}{\partial z}\begin{bmatrix} e_1 \\ e_2 \\ e_r \end{bmatrix} = \underbrace{\widetilde{P}_1\frac{\partial}{\partial z}}_{\mathcal{J}_e}\begin{bmatrix} e_1 \\ e_2 \\ e_r \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial z}(e_2 + e_r) \\ \frac{\partial e_1}{\partial z} \\ \frac{\partial e_1}{\partial z} \end{bmatrix}. \quad (6.51)$$

In this case, neither condition (6.4a) nor (6.4b) is satisfied. This obviously gives that the corresponding matrix Q is singular. In fact, in this case the port-variables would be

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} e_2(b) - e_2(a) + e_r(b) - e_r(a) \\ e_1(b) - e_1(a) \\ e_1(b) - e_1(a) \\ e_1(b) + e_1(a) \\ e_2(b) + e_2(a) \\ e_r(b) + e_r(a) \end{bmatrix}.$$

From this it is clear that the range of $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ cannot be $\mathbb{R}^6$. Furthermore, the non-invertibility of $\widetilde{P}_1$ (equivalently, of Q) gives that selecting $H^N(a,b)^n$ $(N = 1)$ as the domain of $\mathcal{J}_e$ is not enough. Indeed, from (6.51) we can see that the maximal domain of $\mathcal{J}_e$ should be $\{e \in L_2(a,b)^3 \mid e_1, (e_2 + e_r) \in H^1(a,b)\}$, which clearly contains $H^1(a,b)^3$. Note that once $\mathcal{J}_e$ is considered in this larger domain (and P_N is nonsingular), the representation for $\mathcal{J}_e$ in (6.51) is not correct. Strictly speaking, it should be written as $\mathcal{J}_e = \frac{\partial}{\partial z}\widetilde{P}_1(\cdot)$. With an abuse of notation we use either one of the representations. Also, in this maximal domain $\frac{\partial}{\partial z}(e_2 + e_r)$ has to be considered as a single expression, i.e., it is *not* necessarily true that $\frac{\partial}{\partial z}(e_2 + e_r) = \frac{\partial e_2}{\partial z} + \frac{\partial e_r}{\partial z}$. In particular, the operator $\mathcal{J}_e$ should be seen as $\mathcal{J}_e = \left[\begin{smallmatrix} \mathcal{J}\&\mathcal{G}_R \\ -\mathcal{G}_R^*,\ 0 \end{smallmatrix}\right]$, where the operator $\mathcal{J}\&\mathcal{G}_R$ when restricted to $H^N(a,b)^{n+m}$ can be split as $[\mathcal{J},\ \mathcal{G}_R]$. $*$

In order to cover these cases we try to extend Theorem 2.1 (or Theorem 6.2) to include matrices Q which are singular. To do so we study the expression appearing in the right hand side of (2.4). For simplicity let $l_i^T = \left[\begin{array}{ccc} e_i^T(z), \ldots & , \frac{d^{N-1}e_i^T}{dz^{N-1}}(z) \end{array}\right]$ and let π denote the orthogonal projector onto the range[1] of Q, which implies that $I - \pi$ is the orthogonal projector onto the kernel of Q, i.e., $Q(I - \pi) = 0$. Then

$$\langle l_1, Q\, l_2\rangle_{\mathbb{R}^{nN}} = \langle (I - \pi + \pi) l_1, Q(I - \pi + \pi)\, l_2\rangle_{\mathbb{R}^{nN}} = \langle \pi\, l_1, Q\, \pi\, l_2\rangle_{\mathbb{R}^{nN}}.$$

Let r be the rank of Q and M be an $nN \times r$ matrix, whose columns are linearly independent and span the range of Q. Then π can be chosen as $\pi = M(M^TM)^{-1}M^T$. In fact, by choosing the columns of M to be eigenvectors corresponding to nonzero eigenvalues of Q, we get that M^TQM is an $r \times r$ nonsingular diagonal matrix and $M^TM = I$, see [Mey01, §7.5]. Using $\pi = M(M^TM)^{-1}M^T$

[1]Recall that since Q is symmetric we have that ran $(Q) \perp \ker(Q)$, see [Mey01, pp. 408-410].

in the equation above yields

$$\begin{aligned}\langle l_1, Q\, l_2\rangle_{\mathbb{R}^{nN}} &= \left\langle M(M^T M)^{-1} M^T\, l_1, Q\, M(M^T M)^{-1} M^T\, l_2\right\rangle_{\mathbb{R}^{nN}} \\ &= \left\langle (M^T M)^{-1} M^T\, l_1, M^T\, Q\, M(M^T M)^{-1} M^T\, l_2\right\rangle_{\mathbb{R}^r} \\ &= \left\langle M_Q\, l_1, \tilde{Q}\, M_Q\, l_2\right\rangle_{\mathbb{R}^r},\end{aligned}$$

where $\widetilde{Q} = M^T\, Q\, M$ and $M_Q = (M^T M)^{-1} M^T$. Observe that $\widetilde{Q}$ is nonsingular since the columns of M span the range of Q. Following this we can rewrite Theorem 2.1 as follows.

Theorem 6.15: *Assume that the conditions on Theorem* 2.1 *hold. Let r be the rank of Q and let M be an $nN \times r$ matrix whose columns span the range of Q. Then for any two functions $e_1,\ e_2 \in H^N(a,b)^n$ we have*

$$\begin{aligned}&\int_a^b (\mathcal{J} e_1)^T(z)\, e_2(z)\, dz + \int_a^b e_1^T(z)(\mathcal{J} e_2)(z)\, dz \\ &= \left(\begin{bmatrix} e_1^T(z), \dots & , \frac{d^{N-1} e_1^T}{dz^{N-1}}(z) \end{bmatrix} M_Q^T\, \widetilde{Q}\, M_Q \begin{bmatrix} e_2(z) \\ \vdots \\ \frac{d^{N-1} e_2}{dz^{N-1}}(z) \end{bmatrix} \right)_a^b \\ &= \left\langle \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau(e_1), \begin{bmatrix} \widetilde{Q} & 0 \\ 0 & -\widetilde{Q} \end{bmatrix} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau(e_2) \right\rangle_{\mathbb{R}^{2r}}, \qquad (6.52)\end{aligned}$$

where $\widetilde{Q} = M^T\, Q\, M$, $M_Q = (M^T M)^{-1} M^T$, and $\tau(\cdot)$ is the operator described in Definition 2.5.

Remark 6.16. Observe that if the matrix Q is nonsingular, then the matrix M can be selected as the identity matrix. Thus, in this case, $\widetilde{Q} = Q$ and $M_Q = I$. ♣

Observe that now equation (6.52) defines a nondegenerate bilinear form. Obviously, once we have redefined Stokes theorem we also need to modify the definition of the boundary port-variables.

Definition 6.17. Consider the operators $\mathcal{J}$ and $\tau(\cdot)$ as described in Theorem 6.15. Then, the *boundary port-variables* associated with the differential operator $\mathcal{J}$ are the vectors $\tilde{e}_\partial, \tilde{f}_\partial \in \mathbb{R}^r$, defined by

$$\begin{bmatrix} \tilde{f}_{\partial,e} \\ \tilde{e}_{\partial,e} \end{bmatrix} = \widetilde{R}_{\text{ext}} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau(e), \quad e \in H^N(a,b)^n, \qquad (6.53)$$

where $\widetilde{R}_{\text{ext}} \in \mathbb{R}^{2r\times 2r}$ is defined by (see Lemma 2.4)

$$\widetilde{R}_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \widetilde{Q} & -\widetilde{Q} \\ I & I \end{bmatrix}. \tag{6.54}$$

♣

Remark 6.18. Using the definition above and Lemma 2.4 it is easy to see that equation (6.52) can be rewritten as

$$\int_a^b (\mathcal{J}e_1)^T(z)\, e_2(z)\, dz + \int_a^b e_1^T(z)(\mathcal{J}e_2)(z)\, dz = \left\langle \begin{bmatrix} \tilde{f}_{\partial,e_1} \\ \tilde{e}_{\partial,e_1} \end{bmatrix}, \Sigma \begin{bmatrix} \tilde{f}_{\partial,e_2} \\ \tilde{e}_{\partial,e_2} \end{bmatrix} \right\rangle_{\mathbb{R}^{2r}}, \tag{6.55}$$

which shows that the bilinear form (2.14) defined in this way is nondegenerate.♣

Note that equations (6.52) and (6.55) hold for all $e_1,\, e_2 \in H^N(a,b)^n$. However, since P_N (equivalently, Q) is not assumed to be invertible, we would like to consider the space (called the *maximal domain* of $\mathcal{J}$)

$$H(J,(a,b))^n = \{x \in L_2(a,b)^n \ \mid \ \mathcal{J}x \in L_2(a,b)^n\} \tag{6.56}$$

as the domain of the skew-symmetric operator $\mathcal{J}$. This is a Hilbert space when endowed with the norm

$$\|x\|_J^2 = \|x\|^2 + \|\mathcal{J}\,x\|^2. \tag{6.57}$$

(Recall that $\|\cdot\|$ denotes the L_2-norm.) Note that in the case $N = 1$, the maximal domain of $\mathcal{J}$ is simply $H(J,(a,b))^n = \{x \in L_2(a,b)^n \ \mid \ P_1\, x \in H^1(a,b)^n\}$. In the case $N \geq 2$ it is better to represent it as (6.56).

It is clear that if we consider the space $H(J,(a,b))^n$ we need to extend Theorem 6.15, and hence (6.55), to functions $e_1,\, e_2 \in H(J,(a,b))^n$. This implies that the boundary port operator $\left[\begin{smallmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right]$ needs to be extended to $H(J,(a,b))$. This follows from the following theorem.

Theorem 6.19: *Let $\mathcal{J}$ be a formally skew-symmetric operator described by* (2.2)–(2.3) *and consider the space $H(J,(a,b))^n$ defined in* (6.56). *Let r be the rank of Q as described in Theorem* 6.15. *The boundary operator $\left[\begin{smallmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right]$ of Definition* 6.17 *defined on $H^N(a,b)^n$ extends by continuity to a continuous linear map – still denoted by $\left[\begin{smallmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right]$ – from $H(J,(a,b))^n$ onto $\mathbb{R}^{2r}$. Furthermore, Green's identity* (6.55) *holds for all $e_1,\, e_2 \in H(J,(a,b))^n$, that is*

$$\int_a^b (\mathcal{J}e_1)^T(z)\, e_2(z)\, dz + \int_a^b e_1^T(z)(\mathcal{J}e_2)(z)\, dz = \left\langle \begin{bmatrix} \tilde{f}_{\partial,e_1} \\ \tilde{e}_{\partial,e_1} \end{bmatrix}, \Sigma \begin{bmatrix} \tilde{f}_{\partial,e_2} \\ \tilde{e}_{\partial,e_2} \end{bmatrix} \right\rangle_{\mathbb{R}^{2r}} \tag{6.58}$$

holds for all $e_1,\, e_2 \in H(J,(a,b))^n$.

PROOF: The key point in the proof is the denseness of $H^N(a,b)^n$ in $H(J,(a,b))^n$ (with the topology induced by (6.57)). This follows from the denseness of the infinitely differential functions in $H(J,(a,b))^n$, which is proved (in a more general case) in Theorem 8.14. Now, since equation (6.55) holds for all $e_1,\, e_2 \in H^N(a,b)^n$, we can use that to prove the desired result by using a density argument. Thus it is easy to see, by using (6.55), that

$$\begin{aligned}\left|\left\langle \begin{bmatrix} \tilde f_{\partial,e_1} \\ \tilde e_{\partial,e_1}\end{bmatrix}, \Sigma \begin{bmatrix} \tilde f_{\partial,e_2} \\ \tilde e_{\partial,e_2}\end{bmatrix}\right\rangle_{\mathbb{R}^{2r}}\right| &\le \|\mathcal{J}\, e_1\|\, \|e_2\| + \|e_1\|\, \|\mathcal{J}\, e_2\| \\ &\le (\|\mathcal{J}\, e_1\| + \|e_1\|)\,(\|e_2\| + \|\mathcal{J}\, e_2\|) \\ &\le c\, \|e_1\|_J\, \|e_2\|_J, \qquad \forall\, e_1,\, e_2 \in H^N(a,b)^n,\end{aligned}$$

where we used the Cauchy-Schwarz inequality and c is a positive constant. The inequality above implies that the bilinear form $\left\langle \begin{bmatrix} \tilde f_{\partial,e_1} \\ \tilde e_{\partial,e_1}\end{bmatrix}, \Sigma \begin{bmatrix} \tilde f_{\partial,e_2} \\ \tilde e_{\partial,e_2}\end{bmatrix}\right\rangle_{\mathbb{R}^{2r}}$ can be extended by density to all $e_1,\, e_2 \in H(J,(a,b))^n$, see [Aub00, §1.3]. The result follows from this. ■

Example 6.20 To illustrate the idea we continue Example 6.14. In this case we have that the matrix M containing the normalized eigenvectors corresponding to nonzero eigenvalues of $Q = \widetilde{P}_1$ is

$$M = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 1 & 0\end{pmatrix} \quad \Rightarrow \quad \widetilde{Q} = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}, \text{ and } M_Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0\end{pmatrix}.$$

It thus follows that the new port-variables are

$$\begin{bmatrix} \tilde f_{\partial,e} \\ \tilde e_{\partial,e}\end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} e_1(b) - e_1(a) \\ (e_2 + e_r)(b) - (e_2 + e_r)(a) \\ (e_2 + e_r)(b) + (e_2 + e_r)(a) \\ e_1(b) + e_1(a)\end{bmatrix} \qquad e = (e_1,\, e_2,\, e_r).$$

We can conclude that the port-variables for the system with structural damping given by (6.50) are

$$\begin{bmatrix} \tilde g_{f_\partial} \\ \tilde g_{e_\partial}\end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} e_1(b) - e_1(a) \\ \left(e_2 + \frac{\partial e_1}{\partial z}\right)(b) - \left(e_2 + \frac{\partial e_1}{\partial z}\right)(a) \\ \left(e_2 + \frac{\partial e_1}{\partial z}\right)(b) + \left(e_2 + \frac{\partial e_1}{\partial z}\right)(a) \\ e_1(b) + e_1(a)\end{bmatrix},$$

which clearly are the boundary variables needed. Furthermore, in this case we also have $H(J_e,(a,b))^3 = \{e \in L_2(a,b)^3 \mid e_1 \in H^1(a,b), \text{ and } (e_2 + e_r) \in H^1(a,b)\}$. It is worth to remark that one first find the port-variables for $e \in H^N(a,b)^{n+m}$ and then this operator is extended to $H(J_e,(a,b))^{n+m}$. ∗

From all this it is easy to see that the results of Section 6.3 apply to the class of systems (6.1) without the need of assumption (6.4a) or (6.4b) by using the modifications presented in this section.

6.5.1. Systems related to skew-symmetric operators

The results presented in Sections 2.2 and 2.3 were valid under the assumption of the invertibility of the matrix Q, see Assumption 2.2. In this subsection we present some ideas that allow us to extend those results so that Assumption 2.2 can be weakened. In other words, we study systems described by (2.1)–(2.3) and we want to obtain BCS in the sense of Section 1.5. In this case, the boundary port-variables are described in Definition 6.17 and the Dirac structure is similar to that in Chapter 2 with the main difference being the domain of $\mathcal{J}$. The flow and effort spaces are

$$\mathcal{F} = \mathcal{E} = L_2(a,b;\mathbb{R}^n) \times \mathbb{R}^r. \tag{6.59}$$

Hence the bond space $\mathcal{B}$ is $\mathcal{F} \times \mathcal{E}$ and, based on equation (6.58), we endow $\mathcal{B}$ with the canonical symmetric pairing described in (2.14).

Theorem 6.21: *Consider the operators* $\mathcal{J}$ *and* $\begin{bmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{bmatrix}$ *as described in Theorem* 6.19*. Let the boundary port-variables be described as in Definition* 6.17*. Then, the subspace* $D_\mathcal{J}$ *of* $\mathcal{B}$ *defined by*

$$D_\mathcal{J} = \left\{ \begin{bmatrix} f \\ \tilde{f}_{\partial,e} \\ e \\ \tilde{e}_{\partial,e} \end{bmatrix} \in \mathcal{B} \;\middle|\; \begin{array}{c} e \in H(J,(a,b))^n, \; \mathcal{J} e = f, \\ \begin{bmatrix} \tilde{f}_{\partial,e} \\ \tilde{e}_{\partial,e} \end{bmatrix}_{|H^N(a,b)^n} = \widetilde{R}_{\text{ext}} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau(e) \end{array} \right\} \tag{6.60}$$

is a Dirac structure with respect to (2.14)*, where* R_{ext} *is given in Definition* 6.17*.*

PROOF: The proof is based on showing the following inclusions

$$\mathcal{D}_\mathcal{J} \subset \mathcal{D}_\mathcal{J}^\perp, \quad \text{and} \quad \mathcal{D}_\mathcal{J}^\perp \subset \mathcal{D}_\mathcal{J},$$

where the orthogonal complement is with respect to (2.14). The first inclusion follows easily from the definition of the bilinear form and Theorem 6.19. In order to prove the other inclusion let $b_1 = [f^T, f_{\partial,e}^T, e^T, e_{\partial,e}^T]^T \in \mathcal{D}_\mathcal{J}$ and $b_2 = [\phi^T, \phi_\partial^T, \eta^T, \eta_\partial^T]^T \in \mathcal{D}_\mathcal{J}^\perp$. Thus

$$0 = \langle b_1, b_2 \rangle_+ = \langle f, \eta \rangle + \langle e, \phi \rangle - \langle f_{\partial,e}, \eta_\partial \rangle_\mathbb{R} - \langle e_{\partial,e}, \phi_\partial \rangle_\mathbb{R}. \tag{6.61}$$

Since $e \in H(J,(a,b))^n$ we can select any e with compact support strictly included in (a,b) in the equation above, which gives

$$0 = \langle f, \eta \rangle + \langle e, \phi \rangle = \langle \mathcal{J}\, e, \eta \rangle + \langle e, \phi \rangle \quad \forall\, e \in \mathcal{D}(a,b)^n.$$

Since this holds for all e with compact support strictly included in (a,b), we can see by using the derivative in the distributional sense, see Definition 8.3, that

$\langle\langle \mathcal{J}\, e, \eta\rangle\rangle_{\mathcal{D}(a,b)} = -\langle\langle e, \mathcal{J}\,\eta\rangle\rangle_{\mathcal{D}(a,b)}$. Using this in the equation above yields

$$0 = -\langle\langle e, \mathcal{J}\,\eta\rangle\rangle_{\mathcal{D}(a,b)} + \langle\langle e, \phi\rangle\rangle \quad \forall\, e \in \mathcal{D}(a,b)^n.$$

From this we conclude that $\mathcal{J}\,\eta = \phi$ in $\mathcal{D}'(a,b)^n$ (recall that $\mathcal{D}'(a,b)$ is the dual of $\mathcal{D}(a,b)$). Since $\phi \in L_2(a,b)^n$ we can conclude that

$$\mathcal{J}\,\eta = \phi\, \in L_2(a,b)^n \quad \text{and hence} \quad \eta \in H(J,(a,b))^n. \tag{6.62}$$

Using the equation above together with (6.60) in (6.61) gives

$$0 = \langle \mathcal{J}\, e, \eta\rangle + \langle e, \mathcal{J}\,\eta\rangle - \langle f_{\partial,e}, \eta_\partial\rangle_{\mathbb{R}} - \langle e_{\partial,e}, \phi_\partial\rangle_{\mathbb{R}},$$

which by Theorem 6.19 and the definition of the boundary port-variables gives, see also (6.58),

$$\begin{aligned} 0 &= \left\langle \begin{bmatrix} \tilde{f}_{\partial,e} \\ \tilde{e}_{\partial,e} \end{bmatrix}, \Sigma \begin{bmatrix} \tilde{f}_{\partial,\eta} \\ \tilde{e}_{\partial,\eta} \end{bmatrix} \right\rangle_{\mathbb{R}} - \left\langle \tilde{f}_{\partial,e}, \eta_\partial \right\rangle_{\mathbb{R}} - \langle \tilde{e}_{\partial,e}, \phi_\partial\rangle_{\mathbb{R}} \\ &= \left\langle \begin{bmatrix} \tilde{f}_{\partial,e} \\ \tilde{e}_{\partial,e} \end{bmatrix}, \Sigma \begin{bmatrix} \tilde{f}_{\partial,\eta} \\ \tilde{e}_{\partial,\eta} \end{bmatrix} \right\rangle_{\mathbb{R}} - \left\langle \begin{bmatrix} \tilde{f}_{\partial,e} \\ \tilde{e}_{\partial,e} \end{bmatrix}, \Sigma \begin{bmatrix} \phi_\partial \\ \eta_\partial \end{bmatrix} \right\rangle_{\mathbb{R}}. \end{aligned}$$

Since $\left[\begin{smallmatrix} \tilde{f}_{\partial,e} \\ \tilde{e}_{\partial,e} \end{smallmatrix}\right]$ is surjective for all $e \in H^N(a,b)^n$, and hence for all $e \in H(J,(a,b))^n$, we can conclude form the equation above that

$$\begin{bmatrix} \phi_\partial \\ \eta_\partial \end{bmatrix} = \begin{bmatrix} \tilde{f}_{\partial,\eta} \\ \tilde{e}_{\partial,\eta} \end{bmatrix}.$$

This gives the desired result. ∎

Now that we have the Dirac structure it is not too difficult to define boundary control systems. Following Section 2.3 it is easy to see that the operator below is a candidate to be the semigroup generator of the BCS.

Lemma 6.22: *Consider the operator $\mathcal{J}$ on $L_2(a,b)^n$ and $\left[\begin{smallmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right]$ as described in Theorem 6.19. Let r be the rank of Q in Theorem 6.15 (or equivalently, the dimension of $\widetilde{Q}$) and let W be a full rank matrix of size $r \times 2r$ which satisfies $W\Sigma W^T \geq 0$. Define the operator $\mathcal{A}$ and its domain, $D(\mathcal{A})$, as*

$$\mathcal{A}\,x = \mathcal{J}\,x \tag{6.63}$$

and

$$D(\mathcal{A}) = \left\{ x \in L_2(a,b)^n \;\middle|\; x \in H(J,(a,b))^n \text{ and } \begin{bmatrix} \tilde{f}_{\partial,x} \\ \tilde{e}_{\partial,x} \end{bmatrix} \in \ker W \right\}. \tag{6.64}$$

Then the adjoint of $\mathcal{A}$ *equals* $-\mathcal{J}$ *with domain (see* (2.20)*)*

$$D(\mathcal{A}^*) = \left\{ x \in H(J,(a,b))^n \;\middle|\; \begin{bmatrix} \tilde{f}_{\partial,x} \\ \tilde{e}_{\partial,x} \end{bmatrix} \in \ker \begin{bmatrix} -(I+V^T), & I-V^T \end{bmatrix} \right\}. \tag{6.65}$$

♡

PROOF: First recall that W can be written as (2.20) since it satisfies $W\Sigma W^T \geq 0$. To find the adjoint of $\mathcal{A}$ we employ the same ideas used in the proof of Theorem 2.24. We use the defining condition (2.63), i.e.,

$$\langle \mathcal{A}\, y, u\rangle = \langle y, w\rangle \quad w \in L_2(a,b)^n,\ \forall\, y \in D(\mathcal{A}).$$

By applying the same ideas used to prove (6.62), i.e., by selecting $y \in \mathcal{D}(a,b)^n$ and using the derivative in the sense of distributions, one can show that $w = -\mathcal{J}\,u$ and hence u must be an element of $H(J,(a,b))^n$, or equivalently, $D(\mathcal{A}^*) \subset H(J,(a,b))^n$. This allows us to use Theorem 6.19. Using (6.58) we obtain

$$\langle \mathcal{A}y, u\rangle = \langle \mathcal{J}\, y, u\rangle = -\langle y, \mathcal{J}\, u\rangle + \begin{bmatrix} \tilde{f}_{\partial,y} \\ \tilde{e}_{\partial,y} \end{bmatrix}^T \Sigma \begin{bmatrix} \tilde{f}_{\partial,u} \\ \tilde{e}_{\partial,u} \end{bmatrix},$$

where Σ is given in (2.9). Since $\left[\begin{smallmatrix} \tilde{f}_{\partial,y} \\ \tilde{e}_{\partial,y} \end{smallmatrix}\right]$ lies in the kernel of W we get, from equation (2.21), that $\left[\begin{smallmatrix} \tilde{f}_{\partial,y} \\ \tilde{e}_{\partial,y} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I-V \\ -(I+V) \end{smallmatrix}\right] l$ for some $l \in \mathbb{R}^r$. Hence

$$\begin{aligned} \langle \mathcal{A}y, u\rangle &= \langle y, -\mathcal{J}\, u\rangle + l^T \begin{bmatrix} I-V^T, & -(I+V^T) \end{bmatrix} \Sigma \begin{bmatrix} \tilde{f}_{\partial,u} \\ \tilde{e}_{\partial,u} \end{bmatrix} \\ &= \langle y, -\mathcal{J}\, u\rangle + l^T \begin{bmatrix} -(I+V^T), & I-V^T \end{bmatrix} \begin{bmatrix} \tilde{f}_{\partial,u} \\ \tilde{e}_{\partial,u} \end{bmatrix}. \end{aligned}$$

Using the defining condition (2.63) and the fact that the equality above must hold for all $l \in \mathbb{R}^{nN}$, we conclude that

$$\begin{bmatrix} \tilde{f}_{\partial,u} \\ \tilde{e}_{\partial,u} \end{bmatrix} \in \ker \begin{bmatrix} -(I+V^T), & I-V^T \end{bmatrix} \quad \text{and} \quad \mathcal{A}^*\, u = -\mathcal{J}\, u.$$

This conclues the proof. ■

Once we know the adjoint of the operator $\mathcal{A}$ in Lemma 6.22 it is easy to prove that it generates a contraction semigroup. The proof is exactly the same as the proof of Theorem 4.3 of [LZM04]. The proof is established by noticing that $e_1 = e_2 \in D(\mathcal{A})$ in (6.58) gives

$$\langle \mathcal{A}e_1, e_1\rangle = \langle \mathcal{J}e_1, e_1\rangle = \frac{1}{2} \begin{bmatrix} \tilde{f}_{\partial,e_1} \\ \tilde{e}_{\partial,e_1} \end{bmatrix}^T \Sigma \begin{bmatrix} \tilde{f}_{\partial,e_1} \\ \tilde{e}_{\partial,e_1} \end{bmatrix}.$$

Since $\begin{bmatrix} \tilde{f}_{\partial,e_1} \\ \tilde{e}_{\partial,e_1} \end{bmatrix}$ lies in the kernel of W we get, from equation (2.21), that $\begin{bmatrix} \tilde{f}_{\partial,e_1} \\ \tilde{e}_{\partial,e_1} \end{bmatrix} = \begin{bmatrix} I-V \\ -(I+V) \end{bmatrix} l$ for some $l \in \mathbb{R}^r$. Thus the equation above becomes

$$\langle \mathcal{A}e_1, e_1 \rangle = \frac{1}{2} l^T \begin{bmatrix} I - V^T, & -(I+V^T) \end{bmatrix} \Sigma \begin{bmatrix} I - V \\ -(I+V) \end{bmatrix} l = l^T(-I+V^T V)l \leq 0,$$

where it was used $V^T V \leq I$, see (2.20). A similar argument shows that $\langle \mathcal{A}^* e, e \rangle \leq 0$ for all $e \in D(\mathcal{A}^*)$. This proof that $\mathcal{A}$ generates a contraction semigroup. Note that if $V^T V = I$, which corresponds to $W \Sigma W^T = 0$, we have that $\mathcal{A}$ generates a unitary semigroup, see [LZM04, §4] for more details.

Now that we know that $\mathcal{A}$ generates a contraction semigroup we can include the operator $\mathcal{L}$ as we did in Section 2.3.3. In this case the proof of the following result is the same as the proof of Theorem 2.13 with the only difference being that the operator $\mathcal{A}$ is the one described in Lemma 6.22.

Theorem 6.23: *Consider the skew-symmetric operator $\mathcal{J}$ on X (see* (2.33)*) and the operator $\begin{bmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{bmatrix}$ as described in Theorem* 6.19*. Let r be the rank of Q in Theorem* 6.15 *(or equivalently, the dimension of $\widetilde{Q}$) and let W be a full rank matrix of size $r \times 2r$. Define the operator $\mathcal{A}_\mathcal{L}$ and its domain, $D(\mathcal{A}_\mathcal{L})$, as*

$$\mathcal{A}_\mathcal{L}\, x = \mathcal{J} \mathcal{L}\, x \tag{6.66}$$

and

$$D(\mathcal{A}_\mathcal{L}) = \left\{ x \in X \mid \mathcal{L}x \in H(J,(a,b))^n \text{ and } \begin{bmatrix} \tilde{f}_{\partial,\mathcal{L}x} \\ \tilde{e}_{\partial,\mathcal{L}x} \end{bmatrix} \in \ker W \right\}. \tag{6.67}$$

Then $\mathcal{A}_\mathcal{L}$ generates a contraction semigroup $T(t)$, $t \geq 0$, on X if and only if W satisfies $W \Sigma W^T \geq 0$.

Furthermore, $\mathcal{A}$ is the infinitesimal generator of a unitary *semigroup on X if and only if W satisfies $W \Sigma W^T = 0$.*

Furthermore, BCS can be obtained as described below.

Theorem 6.24: *Consider the skew-symmetric operator $\mathcal{J}$ on X (see* (2.33)*) and the operator $\begin{bmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{bmatrix}$ as described in Theorem* 6.21*. Let r be the rank of Q in Theorem* 6.15 *(or equivalently, the dimension of $\widetilde{Q}$) and let W be a full rank matrix of size $r \times 2r$. If W satisfies $W \Sigma W^T \geq 0$, where Σ is defined in* (2.9)*,* *then the following system*

$$\frac{\partial x}{\partial t}(t) = \mathcal{J}\mathcal{L}x(t), \quad \text{or equivalently} \quad \Big(\dot{x}(t),\, \tilde{f}_{\partial,\mathcal{L}x}(t),\, \mathcal{L}x(t),\, \tilde{e}_{\partial,\mathcal{L}x}(t)\Big) \in \mathcal{D}_{\mathcal{J}}$$

defined on the state space X *(see (2.33)) with input*

$$u(t) = \mathfrak{B}\, x(t) = W \begin{bmatrix} \tilde{f}_{\partial,\mathcal{L}x}(t) \\ \tilde{e}_{\partial,\mathcal{L}x}(t) \end{bmatrix}$$

is a boundary control system on X*. Furthermore, the operator* $\mathcal{A}_{\mathcal{L}} = \mathcal{J}\mathcal{L}$ *with domain*

$$D(\mathcal{A}_{\mathcal{L}}) = \left\{ x \in X \;\middle|\; \mathcal{L}x \in H(J,(a,b))^n \text{ and } \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} \in \ker W \right\}, \tag{6.68}$$

generates a contraction semigroup on X*.*

Let $\widetilde{W}$ *be a full rank matrix of size* $r \times 2r$ *such that* $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ *is invertible. If we define the linear mapping* $\mathcal{C} : \mathcal{L}^{-1}H(J,(a,b))^n \to \mathbb{R}^r$ *as,*

$$\mathcal{C}x(t) := \widetilde{W} \begin{bmatrix} \tilde{f}_{\partial,\mathcal{L}x}(t) \\ \tilde{e}_{\partial,\mathcal{L}x}(t) \end{bmatrix} \tag{6.69}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{6.70}$$

then for $u \in C^2(0,\infty;\mathbb{R}^r)$*,* $\mathcal{L}x(0) \in H(J,(a,b))^n$*, and* $\mathcal{B}x(0) = u(0)$ *the following balance equation is satisfied:*

$$\frac{d}{dt}H(t) = \frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = \frac{1}{2} \begin{bmatrix} u^T(t) & y^T(t) \end{bmatrix} P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \tag{6.71}$$

where

$$P^{-1}_{W,\tilde{W}} = \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix} \Sigma \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^T = \begin{bmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{bmatrix}. \tag{6.72}$$

Furthermore, we have that the matrix $\left(\begin{smallmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{smallmatrix}\right)$ *is invertible if and only if* $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ *is invertible.*

These results can also be used to prove similar results for systems with dissipation as it is shown in the next subsection.

Remark 6.25. Even though we have extended the results of Sections 2.2 and 2.3 to a larger class of systems, it does not necessarily follow that the results of Section 2.5 hold for this larger class. For instance, it is know that the vibrating string with structural damping of Example 6.14 does not have a compact resolvent, see [LT98]. ♣

6.5.2. Systems with dissipation

Now that we have the results for skew-symmetric operators we can extend the results presented in Section 6.3. That is we study systems described by (6.1)–(6.2) were it is only assumed condition (6.3). As we know from Section 6.1 there is a skew-symmetric operator $\mathcal{J}_e$ related to the class of systems (6.1)–(6.2). Thus we only need to use the results of the previous subsection to obtain BCS from this class of systems (as we did in Section 6.3). First we state the results for the extended operator $\mathcal{J}_e$ which is related to $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$.

Systems related to $\mathcal{J}_e$

In this subsection we consider the operator $\mathcal{J}_e$ described in Section 6.1. However, since we extend this operator to the space $H(J_e,(a,b))^{n+m}$, see (6.56), we have to consider

$$\mathcal{J}_e = \begin{bmatrix} \multicolumn{2}{c}{\mathcal{J}\&\mathcal{G}_R} \\ -\mathcal{G}_R^* & 0 \end{bmatrix}, \quad D(\mathcal{J}_e) = H(J_e,(a,b))^{n+m}, \tag{6.73}$$

where the operator $\mathcal{J}\&\mathcal{G}_R$ when restricted to $H^N(a,b)^{n+m}$ can be seen as the two operators $\begin{bmatrix} \mathcal{J} & \mathcal{G}_R \end{bmatrix}$. First we need to redefine the boundary port-variables based on Definition 6.17 and the ideas of Section 6.1, see equation (6.23).

Definition 6.26. Consider the operator $\mathcal{J}_e$ related to $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$. Let r be the dimension of $\widetilde{Q}$ related to $\mathcal{J}_e$ as in Theorem 6.15 and $\tau : H^N(a,b)^{n+m} \rightarrow \mathbb{R}^{2(n+m)N}$ be given by (2.10). Then, the *boundary port-variables* associated with $\mathcal{J}_e$ are the vectors $\tilde{f}_\partial, \tilde{e}_\partial \in \mathbb{R}^r$, defined by

$$\begin{bmatrix} \tilde{f}_{\partial,\tilde{e}} \\ \tilde{e}_{\partial,\tilde{e}} \end{bmatrix} = \widetilde{R}_{\text{ext}} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau\left(\begin{bmatrix} e \\ e_r \end{bmatrix}\right), \quad \text{with } \tilde{e} = \begin{bmatrix} e \\ e_r \end{bmatrix}, \tag{6.74}$$

where $\widetilde{R}_{\text{ext}} \in \mathbb{R}^{2r \times 2r}$ is defined by (see Lemma 2.4)

$$\widetilde{R}_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \widetilde{Q} & -\widetilde{Q} \\ I & I \end{bmatrix}. \tag{6.75}$$

♣

It is clear now, from Theorem 6.19, that these port-variables can also be extended to $H(J_e,(a,b))^{n+m}$. Also the operator $\mathcal{J}_e$ defines a Dirac structure, see Theorem 6.21. Now that we have defined the boundary port-variables and the Dirac structure which are related to the extended operator $\mathcal{J}_e$ on (6.73) we can rewrite Theorem 2.14 adapted to this skew-symmetric operator.

In other words, we consider systems described by the following PDE

$$\frac{\partial x}{\partial t}(t) = \mathcal{J}_e \tilde{\mathcal{L}} x(t),$$

where $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L}_1 & 0 \\ 0 & I \end{smallmatrix}\right]$ is a coercive operator on the state space $\tilde{X}$ defined in (6.17) and define BCS. Following Theorem 6.24 it is easy to see that the following results holds.

Theorem 6.27: *Consider the skew-symmetric operator $\mathcal{J}_e$ in (6.73). Let r be the rank of Q in Theorem 6.15 (or equivalently, the dimension of $\widetilde{Q}$) and let W be a full rank matrix of size $r \times 2r$. If W satisfies $W\Sigma W^T \geq 0$, where Σ is defined in (2.9), then the following system with $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L}_1 & 0 \\ 0 & I \end{smallmatrix}\right]$*

$$\frac{\partial x}{\partial t}(t) = \mathcal{J}_e \tilde{\mathcal{L}} x(t), \quad \text{or equivalently} \quad \left(\dot{x}(t),\, \tilde{f}_{\partial,\tilde{\mathcal{L}}x}(t),\, \tilde{\mathcal{L}}x(t),\, \tilde{e}_{\partial,\tilde{\mathcal{L}}x}(t)\right) \in \mathcal{D}_{\mathcal{J}_e}$$

defined on the state space $\tilde{X}$ (see (6.17)) with input

$$u(t) = \mathfrak{B}\, x(t) = W \begin{bmatrix} \tilde{f}_{\partial,\tilde{\mathcal{L}}x}(t) \\ \tilde{e}_{\partial,\tilde{\mathcal{L}}x}(t) \end{bmatrix}$$

is a boundary control system on $\tilde{X}$. Furthermore, the operator $\mathcal{A}_{\text{ext}} = \mathcal{J}_e \tilde{\mathcal{L}}$ with domain

$$D(\mathcal{A}_{\text{ext}}) = \left\{ \tilde{x} = \begin{bmatrix} x \\ x_r \end{bmatrix} \in \tilde{X} \mid \tilde{\mathcal{L}}\tilde{x} \in H(J_e,(a,b))^{n+m} \text{ and } \begin{bmatrix} \tilde{f}_{\partial,\tilde{\mathcal{L}}\tilde{x}} \\ \tilde{e}_{\partial,\tilde{\mathcal{L}}\tilde{x}} \end{bmatrix} \in \ker W \right\} \tag{6.76}$$

generates a contraction semigroup on $\tilde{X}$.

Let $\widetilde{W}$ be a full rank matrix of size $r \times 2r$ such that $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ is invertible. If we define the linear mapping $\mathcal{C} : \mathcal{L}^{-1} H(J_e,(a,b))^{n+m} \to \mathbb{R}^r$ as,

$$\mathcal{C}x(t) := \widetilde{W} \begin{bmatrix} \tilde{f}_{\partial,\tilde{\mathcal{L}}x}(t) \\ \tilde{e}_{\partial,\tilde{\mathcal{L}}x}(t) \end{bmatrix} \tag{6.77}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{6.78}$$

then for $u \in C^2(0,\infty;\mathbb{R}^r)$, $\tilde{\mathcal{L}}x(0) \in H(J_e,(a,b))^{n+m}$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\tilde{\mathcal{L}}} = \frac{1}{2} \begin{bmatrix} u^T(t) & y^T(t) \end{bmatrix} P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \tag{6.79}$$

where

$$P^{-1}_{W,\tilde{W}} = \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix} \Sigma \begin{bmatrix} W \\ \widetilde{W} \end{bmatrix}^T = \begin{bmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{bmatrix}. \tag{6.80}$$

Furthermore, we have that the matrix $\begin{pmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{pmatrix}$ *is invertible if and only if* $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ *is invertible.*

Now that we have defined BCS for the operator $\mathcal{J}_e$ we can proceed to define BCS for the class of systems (6.1).

Systems related to $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}$

Now we turn to systems described by (6.1) where it is only assumed that (6.3) holds. Recall that the state space X is defined in (2.33) (note that it is different from $\tilde{X}$).

Before stating the main result, we stress that if we define (with $\mathcal{J}_e$ in (6.73))

$$\begin{bmatrix} f \\ f_r \end{bmatrix} = \mathcal{J}_e \begin{bmatrix} e \\ e_r \end{bmatrix} = \begin{bmatrix} \mathcal{J}\&\mathcal{G}_R \\ -\mathcal{G}_R^* \quad 0 \end{bmatrix} \begin{bmatrix} e \\ e_r \end{bmatrix}$$

and let $e_r = Sf_r = -S\mathcal{G}_R^* e$ with S a coercive operator, see Figure 6.1, we obtain

$$f = \mathcal{J}\&\mathcal{G}_R \begin{bmatrix} e \\ -S\mathcal{G}_R^* e \end{bmatrix}$$

which is the same operator that defines our class of systems (6.1), see Section 6.1. As mentioned earlier, this idea of feedback will be used to prove the main results of this subsection. First, based on Definition 6.26, we introduce again the notion of boundary port-variables adapted to the class of systems described by (6.1). Then we proceed to prove the main results. For simplicity we denote by $H(A_g, (a,b))^n$ the space described by

$$H(A_g, (a,b))^n = \left\{ x \in X \,\middle|\, \begin{bmatrix} \mathcal{L}x \\ -S\mathcal{G}_R^*\mathcal{L}x \end{bmatrix} \in H(J_e, (a,b))^{n+m} \right\}, \tag{6.81}$$

where $\mathcal{J}_e$ is the skew-symmetric operator related to $\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*$, i.e., $\mathcal{J}_e = \left[\begin{smallmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{smallmatrix}\right]$, see Section 6.1. This can be considered as the maximal domain of the operator $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$.

Definition 6.28. Consider the operator $\mathcal{J}_e$ described in (6.73) related to $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$. Let r be the dimension of $\widetilde{Q}$ related to $\mathcal{J}_e$ as in Theorem 6.15 and

$\tau : H^N(a,b)^{n+m} \rightarrow \mathbb{R}^{2(n+m)N}$ be given by (2.10). Then, the *boundary port-variables* associated with the differential operator $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$ are given by the operator $\left[\begin{smallmatrix} \tilde{g}_{e\partial} \\ \tilde{g}_{f\partial} \end{smallmatrix}\right] : H(A_g, (a,b))^n \rightarrow \mathbb{R}^{2r}$ defined (when $e \in H^N(a,b)^n$) by

$$\begin{bmatrix} \tilde{g}_{f\partial,e} \\ \tilde{g}_{e\partial,e} \end{bmatrix}_{|H^N(a,b)^{n+m}} = \widetilde{R}_{\text{ext}} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau \left(\begin{bmatrix} e \\ -S\mathcal{G}_R^* e \end{bmatrix} \right), \tag{6.82}$$

where $\widetilde{R}_{\text{ext}} \in \mathbb{R}^{2r \times 2r}$ is defined by (see Lemma 2.4)

$$\widetilde{R}_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \widetilde{Q} & -\widetilde{Q} \\ I & I \end{bmatrix}. \tag{6.83}$$

♣

Now it is easy to see that the results of Section 6.3.2 also hold in this case if we use the definition above. In particular, we have the following two theorems.

Theorem 6.29: *Let r be the dimension of $\tilde{g}_{f\partial}$ and $\tilde{g}_{e\partial}$ according to Definition 6.28 and let W be an $r \times 2r$ matrix. Consider the operator $\mathcal{J}_e$ related to $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$ and the operator $\mathcal{A}_g = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}$ with domain*

$$\begin{aligned} D(\mathcal{A}_g) &= \left\{ x \in X \,\middle|\, \begin{bmatrix} x \\ -S\mathcal{G}_R^* \mathcal{L}x \end{bmatrix} \in D(\mathcal{A}_{\text{ext}}) \right\} \\ &= \left\{ x \in X \,\middle|\, \begin{bmatrix} \mathcal{L}x \\ -S\mathcal{G}_R^* \mathcal{L}x \end{bmatrix} \in H(J_e, (a,b))^{n+m}, \begin{bmatrix} \tilde{g}_{f\partial,\mathcal{L}x} \\ \tilde{g}_{e\partial,\mathcal{L}x} \end{bmatrix} \in \ker W \right\}. \end{aligned} \tag{6.84}$$

If W has full rank and satisfies $W \Sigma W^T \geq 0$, where Σ is given by (2.9), then $\mathcal{A}_g$ generates a contraction semigroup on X and it satisfies

$$\langle \mathcal{A}_g \, x, x \rangle_{\mathcal{L}} \leq - \langle \mathcal{G}_R^* \mathcal{L}x, S\mathcal{G}_R^* \mathcal{L}x \rangle, \quad \forall \, x \in D(\mathcal{A}_g). \tag{6.85}$$

Remark 6.30. Note that for simplicity we represent $\mathcal{A}_g$ as $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}$, but strictly speaking, in this case, it should be written as $\mathcal{A}_g \, x = \mathcal{J} \& \mathcal{G}_R \left[\begin{smallmatrix} \mathcal{L}x \\ -S\mathcal{G}_R^* \mathcal{L}x \end{smallmatrix}\right]$. ♣

PROOF (PROOF OF THEOREM 6.29): The proof is similar to the proof of Theorem 6.9 and it is based on a feedback argument on the operator $\mathcal{J}_e$. We make use of its corresponding semigroup generator $\mathcal{A}_{\text{ext}}$ described in Theorem 6.27. Since $\mathcal{A}_{\text{ext}}$ is the generator of a contraction semigroup (see Theorem 6.27) on $\tilde{X}$ (see (6.17)) we have from the Lümer-Phillips theorem (see [Paz83, §1.4 and §3.3]) that

$$\langle \mathcal{A}_{\text{ext}} \tilde{x}, \tilde{x} \rangle_{\tilde{\mathcal{L}}} \leq 0 \quad \text{for all } \tilde{x} \in D(\mathcal{A}_{\text{ext}}) \text{ and} \tag{6.86}$$

$$\text{ran}\,(\lambda I - \mathcal{A}_{\text{ext}}) = \tilde{X} \quad \text{for some } \lambda > 0. \tag{6.87}$$

Now we can proceed to prove that $\mathcal{A}_g$ generates a contraction semigroup on X. To do so, we use the same Lümer-Phillips theorem. That is, we first prove that $\mathcal{A}_g$ satisfies $\langle \mathcal{A}_g\, x, x\rangle_{\mathcal{L}} \leq 0$ for any $x \in D(\mathcal{A}_g)$ and next that ran $(\lambda I - \mathcal{A}_g) = X$ for some $\lambda > 0$. For $x \in D(\mathcal{A}_g)$, we have

$$\langle \mathcal{A}_g\, x, x\rangle_{\mathcal{L}} = \left\langle \mathcal{J}\&\mathcal{G}_R \begin{bmatrix} \mathcal{L}x \\ -S\mathcal{G}_R^*\mathcal{L}x \end{bmatrix}, \mathcal{L}x \right\rangle$$

Define $x_r = -S\mathcal{G}_R^*\mathcal{L}x$ and observe that $\left[\begin{smallmatrix} x \\ x_r \end{smallmatrix}\right] \in D(\mathcal{A}_{\text{ext}})$ since $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L} & 0 \\ 0 & I \end{smallmatrix}\right]$, see (6.76) and Definitions 6.26 and 6.28. From this, the equation above, and since $\tilde{\mathcal{L}} = \left[\begin{smallmatrix} \mathcal{L} & 0 \\ 0 & I \end{smallmatrix}\right]$ we can see that

$$\begin{aligned}
\langle \mathcal{A}_g\, x, x\rangle_{\mathcal{L}} &= \left\langle \mathcal{J}\&\mathcal{G}_R \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix}, \mathcal{L}x \right\rangle \\
&= \left\langle \mathcal{J}\&\mathcal{G}_R \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix}, \mathcal{L}x \right\rangle + \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \\
&= \left\langle \mathcal{J}\&\mathcal{G}_R \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix}, \mathcal{L}x \right\rangle + \langle \mathcal{G}_R^*\mathcal{L}x, -x_r\rangle - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \\
&= \left\langle \begin{bmatrix} \mathcal{J}\&\mathcal{G}_R \\ -\mathcal{G}_R^* \quad 0 \end{bmatrix} \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix}, \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix} \right\rangle - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \\
&= \left\langle \mathcal{A}_{\text{ext}} \begin{bmatrix} x \\ x_r \end{bmatrix}, \begin{bmatrix} x \\ x_r \end{bmatrix} \right\rangle_{\tilde{\mathcal{L}}} - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \\
&\leq - \langle \mathcal{G}_R^*\mathcal{L}x, S\mathcal{G}_R^*\mathcal{L}x\rangle \leq 0,
\end{aligned} \tag{6.88}$$

where in the third step we used $x_r = -S\mathcal{G}_R^*\mathcal{L}x$ and in the last step we used (6.86) and the fact that S is coercive, see (6.5).

Next we prove the range condition on $\mathcal{A}_g$. That is, for a $\lambda > 0$ we have to show that for any given $f \in X$ we can find an $x \in D(\mathcal{A}_g)$ such that

$$f = (\lambda I - \mathcal{A}_g)x.$$

In order to prove this, let

$$P = \begin{bmatrix} 0 & 0 \\ 0 & S^{-1} - \lambda I \end{bmatrix}.$$

Since S is coercive, we can find some $\lambda > 0$ such that $S^{-1} - \lambda I \geq 0$. Thus we can assume that P is a nonnegative operator. It thus follows from Corollary 3.3 of [Paz83] that $\mathcal{A}_{\text{ext}} - P$ generates a contraction semigroup. This in turn implies (by the Lümer-Phillips theorem) that ran $(\lambda I - \mathcal{A}_{\text{ext}} + P) = \tilde{X}$. Thus, given any

$\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] \in \tilde{X}$ we can find $\left[\begin{smallmatrix} x \\ x_r \end{smallmatrix}\right] \in D(\mathcal{A}_{\text{ext}})$ such that

$$\begin{bmatrix} f \\ 0 \end{bmatrix} = (\lambda I - \mathcal{A}_{\text{ext}} + P) \begin{bmatrix} x \\ x_r \end{bmatrix} = \left(\lambda \begin{bmatrix} \mathcal{L}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\mathcal{J}\&\mathcal{G}_R \\ \mathcal{G}_R^* \quad S^{-1} \end{bmatrix} \right) \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix}$$

$$\Rightarrow \quad f = \lambda\, x - \mathcal{J}\&\mathcal{G}_R \begin{bmatrix} \mathcal{L}x \\ x_r \end{bmatrix} \quad \text{and} \quad x_r = -S\mathcal{G}_R^*\mathcal{L}x$$

$$\Rightarrow \quad f = \lambda\, x - \mathcal{J}\&\mathcal{G}_R \begin{bmatrix} \mathcal{L}x \\ -S\mathcal{G}_R^*\mathcal{L}x \end{bmatrix}. \tag{6.89}$$

Since $\left[\begin{smallmatrix} x \\ x_r \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ -S\mathcal{G}_R^*\mathcal{L}x \end{smallmatrix}\right] \in D(\mathcal{A}_{\text{ext}})$, it is easy to see that $x \in D(\mathcal{A}_g)$. Then, from (6.89) we can see that $\mathcal{A}_g$ satisfies the range condition. Concluding, we see that $\mathcal{A}_g$ generates a contraction semigroup. ∎

Once we have parameterized the set of boundary conditions for which $\mathcal{A}_g$ generates a contraction semigroup we can define boundary control systems easily. Using this it is easy to see that the proof of the theorem below follow the same ideas as the proof of Theorem 6.11. Also note that $D(\mathcal{A}_g) \subset H(A_g, (a,b))^n$ and that is why $H(A_g, (a,b))^n$ was defined as the maximal domain where the boundary port operator $\left[\begin{smallmatrix} \tilde{g}_{f\partial} \\ \tilde{g}_{e\partial} \end{smallmatrix}\right]$ is well-defined.

Theorem 6.31: *Let r be the dimension of $\tilde{g}_{f\partial}$ and $\tilde{g}_{e\partial}$ according to Definition 6.28, and let W be a $r \times 2r$ matrix. If W has full rank and satisfies $W\Sigma W^T \geq 0$, where Σ is defined in (2.9), then the system (see Remark 6.30)*

$$\frac{\partial x}{\partial t}(t) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}x(t) \tag{6.90}$$

with input

$$u(t) = \mathcal{B}x(t) = W \begin{bmatrix} \tilde{g}_{f\partial,\mathcal{L}x}(t) \\ \tilde{g}_{e\partial,\mathcal{L}x}(t) \end{bmatrix} \tag{6.91}$$

is a boundary control system on X. Furthermore, the operator $\mathcal{A}_g = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}$ with domain

$$D(\mathcal{A}_g) = \left\{ x \in X \,\middle|\, \begin{bmatrix} \mathcal{L}x \\ -S\mathcal{G}_R^*\mathcal{L}x \end{bmatrix} \in H(J_e, (a,b))^{n+m}, \begin{bmatrix} \tilde{g}_{f\partial,\mathcal{L}x} \\ \tilde{g}_{e\partial,\mathcal{L}x} \end{bmatrix} \in \ker W \right\}. \tag{6.92}$$

generates a contraction semigroup on X.

Let $\widetilde{W}$ be a full rank matrix of size $r \times 2r$ with $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$ invertible. If we define the linear mapping $\mathcal{C} : H(A_g, (a,b))^n \to \mathbb{R}^r$ as,

$$\mathcal{C}x(t) := \widetilde{W} \begin{bmatrix} \tilde{g}_{f\partial,\mathcal{L}x}(t) \\ \tilde{g}_{e\partial,\mathcal{L}x}(t) \end{bmatrix} \tag{6.93}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{6.94}$$

then for $u \in C^2(0,\infty;\mathbb{R}^r)$*,* $\mathcal{L}x(0) \in H(A_g,(a,b))^n$*, and* $\mathcal{B}x(0) = u(0)$ *the following balance equation is satisfied:*

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 &= \frac{1}{2}\begin{bmatrix} u^T(t) & y^T(t) \end{bmatrix} P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} - \langle \mathcal{G}_R^* \mathcal{L}x(t), S\mathcal{G}_R^*\mathcal{L}x(t)\rangle \\ &\leq \frac{1}{2}\begin{bmatrix} u^T(t) & y^T(t) \end{bmatrix} P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \end{aligned} \tag{6.95}$$

where

$$P_{W,\tilde{W}}^{-1} = \begin{bmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{bmatrix}. \tag{6.96}$$

Furthermore, the invertibility of $\begin{pmatrix} W\Sigma W^T & W\Sigma\widetilde{W}^T \\ \widetilde{W}\Sigma W^T & \widetilde{W}\Sigma\widetilde{W}^T \end{pmatrix}$ *is equivalent to the invertibility of* $\left[\begin{smallmatrix} W \\ \widetilde{W} \end{smallmatrix}\right]$*.*

Example 6.32 (Swelling porous elastic soils) The model studied here provides the basis for the treatment of various practical problems in the field of swelling, oil explanations, slurred and consolidation problems, see [Eri94]. This formulation belongs to a mix of theories for porous elastic solids filled with fluid. Heat conduction is included. The field equations of the linear theory of swelling porous elastic soils in the case of fluid saturation are (see [Qui02])

$$\begin{aligned} \rho_w \frac{\partial^2 w}{\partial t^2} &= a_1 \frac{\partial^2 w}{\partial z^2} + a_2 \frac{\partial^2 u}{\partial z^2} + \beta_1 \frac{\partial T}{\partial z} - \xi\left(\frac{\partial w}{\partial t} - \frac{\partial u}{\partial t}\right) + \mu_w \frac{\partial^3 w}{\partial z^2 \partial t} \\ \rho_u \frac{\partial^2 u}{\partial t^2} &= a_2 \frac{\partial^2 w}{\partial z^2} + a_3 \frac{\partial^2 u}{\partial z^2} + \beta_2 \frac{\partial T}{\partial z} + \xi\left(\frac{\partial w}{\partial t} - \frac{\partial u}{\partial t}\right) \\ c\frac{\partial T}{\partial t} &= \beta_1 \frac{\partial^2 w}{\partial z \partial t} + \beta_2 \frac{\partial^2 u}{\partial z \partial t} + k\frac{\partial^2 T}{\partial z^2}, \end{aligned} \tag{6.97}$$

where w and u represent the displacements of fluid and solid elastic materials at space position $z \in (a;b)$ and time $t > 0$, respectively. T represents the temperature. The constants $\rho_w,\ \rho_u > 0$ are the densities of each constituent and c is the heat capacity. The parameters $a_1,\ a_2,\ a_3,\ \beta_1,\ \beta_2, \xi,\ \mu_w$ and k are the constitutive constants. The energy function can be described by

$$\begin{aligned} E(t) = \frac{1}{2}\int_a^b \Bigg[& \rho_w \left(\frac{\partial w}{\partial t}\right)^2 + \rho_u \left(\frac{\partial u}{\partial t}\right)^2 \\ & + a_1 \left(\frac{\partial w}{\partial z}\right)^2 + 2a_2 \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} + a_3 \left(\frac{\partial u}{\partial z}\right)^2 + cT^2 \Bigg] dz. \end{aligned} \tag{6.98}$$

This model can be written as an evolution equation by selecting the state variables

$$x_1 = \frac{\partial w}{\partial z}, \quad x_2 = \rho_w \frac{\partial w}{\partial t}, \quad x_3 = \frac{\partial u}{\partial z}, \quad x_4 = \rho_u \frac{\partial u}{\partial t}, \quad x_5 = cT.$$

Using this, equations (6.97) can be rewritten as (with an abuse of notation)

$$\frac{\partial}{\partial t} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & \beta_2 \\ 0 & \beta_1 & 0 & \beta_2 & 0 \end{bmatrix} \frac{\partial}{\partial z}}_{\mathcal{J}} \begin{bmatrix} a_1 x_1 + a_2 x_3 \\ \rho_w^{-1} x_2 \\ a_2 x_1 + a_3 x_3 \\ \rho_u^{-1} x_4 \\ c^{-1} x_5 \end{bmatrix}$$
$$- \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & (-\mu_w \partial_z^2 + \xi) & 0 & -\xi & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\xi & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & -k\partial_z^2 \end{bmatrix}}_{\mathcal{G}_R S \mathcal{G}_R^*} \underbrace{\begin{bmatrix} a_1 x_1 + a_2 x_3 \\ \rho_w^{-1} x_2 \\ a_2 x_1 + a_3 x_3 \\ \rho_u^{-1} x_4 \\ c^{-1} x_5 \end{bmatrix}}_{\mathcal{L} x}. \tag{6.99}$$

From this we can see that

$$\mathcal{G}_R = \begin{bmatrix} 0 & 0 & 0 \\ \partial_z & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \partial_z \end{bmatrix}, \; S = \begin{bmatrix} \mu_w & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & k \end{bmatrix}, \; \mathcal{G}_R^* = \begin{bmatrix} 0 & -\partial_z & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\partial_z \end{bmatrix},$$

and

$$\mathcal{L} = \begin{bmatrix} a_1 & 0 & a_2 & 0 & 0 \\ 0 & \rho_w^{-1} & 0 & 0 & 0 \\ a_2 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & \rho_u^{-1} & 0 \\ 0 & 0 & 0 & 0 & c^{-1} \end{bmatrix}.$$

Typically, it is assumed that $\left[\begin{smallmatrix} a_1 & a_2 \\ a_2 & a_3 \end{smallmatrix}\right]$ is positive definite, see [WG06] or [Qui02], so that $\mathcal{L}$ is coercive. Also note that $N = 1$, $n = 5$, $m = 3$, $r = 6$. In this case we have that the extended operator $\mathcal{J}_e = \left[\begin{smallmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{smallmatrix}\right]$ is given by

$$\mathcal{J}_e\, e_e = \begin{bmatrix} \partial_z e_2 \\ \partial_z(e_1+e_6)+\beta_1\partial_z e_5 - e_7 \\ \partial_z e_4 \\ \partial_z e_3 + \beta_2 \partial_z e_5 + e_7 \\ \beta_1 \partial_z e_2 + \beta_2 \partial_z e_4 + \partial_z e_8 \\ \partial_z e_2 \\ e_2 - e_4 \\ \partial_z e_5 \end{bmatrix}, \quad e_e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{bmatrix}. \tag{6.100}$$

From this it is easy to see that the space $H(A_g,(a,b))^5$ in (6.81) is given by

$$\left\{ x \in X \,\middle|\, \begin{array}{c} (\rho_w^{-1}x_2),\ (\rho_u^{-1}x_4),\ (c^{-1}x_5),\ \big(k\partial_z(c^{-1}x_5)\big) \in H^1(a,b); \\ (a_2\,x_1 + a_3x_3),\ \big(a_1\,x_1 + a_2x_3 + \mu_w\partial_z(\rho_w^{-1}x_2)\big) \in H^1(a,b) \end{array} \right\}. \tag{6.101}$$

The corresponding matrices Q and $\widetilde{Q}$ appearing in Theorem 6.15 are

$$Q = \left[\begin{array}{ccccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \beta_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \beta_2 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \beta_2 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \Rightarrow \widetilde{Q} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & \beta_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \beta_2 & 0 \\ 0 & \beta_1 & 0 & \beta_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

with $M = [\mathbf{v}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4,\, \mathbf{e}_5,\, \mathbf{e}_8]$, where $\mathbf{v}_1 = [1,\, 0,\, 0,\, 0,\, 0,\, 1,\, 0,\, 0]^T$ and $\mathbf{e}_i \in \mathbb{R}^8$, $i = \{2,3,4,5,8\}$, is the vector containing 1 in its i-th component and zero elsewhere. Thus M_Q is given by, see Theorem 6.15,

$$M_Q = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next we proceed to define the boundary port-variables. First notice that

$$-S\mathcal{G}_R^*\mathcal{L}x = \begin{bmatrix} \mu_w\partial_z(\rho_w^{-1}x_2) \\ \xi(\rho_w^{-1}x_2 - \rho_u^{-1}x_4) \\ k\partial_z(c^{-1}x_5) \end{bmatrix},$$

and that the second component of $-S\mathcal{G}_R^*\mathcal{L}x$ is dropped when $\begin{bmatrix} \mathcal{L}x \\ -S\mathcal{G}_R^*\mathcal{L}x \end{bmatrix}$ is pre-multiplied by M_Q. This is because this component is not needed in the definition of the port-variables, since it is already included in $\mathcal{L}x$. For this selection of M the port-variables are (see Definition 6.28)

$$\begin{bmatrix} \tilde{g}_{f\partial,e} \\ \tilde{g}_{e\partial,e} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2e_2(b)-2e_2(a) \\ \big(e_1+\mu_w\partial_z(e_2)\big)(b)+\beta_1e_5(b)-\big(e_1+\mu_w\partial_z(e_2)\big)(a)-\beta_1e_5(a) \\ e_4(b)-e_4(a) \\ e_3(b)+\beta_2e_5(b)-e_3(a)-\beta_2e_5(a) \\ \beta_1e_2(b)+\beta_2e_4(b)+k\partial_z(e_5)(b)-\beta_1e_2(a)-\beta_2e_4(a)-k\partial_z(e_5)(a) \\ e_5(b)-e_5(a) \\ \frac{1}{2}\big(e_1+\mu_w\partial_z(e_2)\big)(b)+\frac{1}{2}\big(e_1+\mu_w\partial_z(e_2)\big)(a) \\ e_2(b)+e_2(a) \\ e_3(b)+e_3(a) \\ e_4(b)+e_4(a) \\ e_5(b)+e_5(a) \\ k\partial_z(e_5)(b)+k\partial_z(e_5)(a) \end{bmatrix}, \tag{6.102}$$

where $e = \mathcal{L}\, x$. Typical boundary conditions for this system are

$$\frac{\partial w}{\partial t}(a) = \frac{\partial w}{\partial t}(b) = 0, \ \frac{\partial u}{\partial t}(a) = \frac{\partial u}{\partial t}(b) = 0, \ T(a) = T(b) = 0,$$

or equivalently $(\rho_w^{-1} x_2)(a) = (\rho_w^{-1} x_2)(b) = 0$, $(\rho_u^{-1} x_4)(a) = (\rho_u^{-1} x_4)(b) = 0$, and $(c^{-1} x_5)(a) = (c^{-1} x_5)(b) = 0$. A corresponding W which gives the boundary conditions (inputs) is given by

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad W \Sigma W^T = 0.$$

With this matrix we obtain

$$W \begin{bmatrix} \tilde{g}_{f\partial,e} \\ \tilde{g}_{e\partial,e} \end{bmatrix} = \begin{bmatrix} \rho_w^{-1} x_2(b) \\ \rho_u^{-1} x_4(b) \\ c^{-1} x_5(b) \\ \rho_w^{-1} x_2(a) \\ \rho_u^{-1} x_4(a) \\ c^{-1} x_5(a) \end{bmatrix}.$$

Note that in this case the domain of the semigroup generator $\mathcal{A}_g$ (see Theorem 6.31 and equations (6.99) and (6.101)) is given by

$$D(\mathcal{A}_g) = \left\{ x \in H(A_g, (a,b)^5 \;\middle|\; \begin{bmatrix} \rho_w^{-1} x_2(b) \\ \rho_u^{-1} x_4(b) \\ c^{-1} x_5(b) \end{bmatrix} = \begin{bmatrix} \rho_w^{-1} x_2(a) \\ \rho_u^{-1} x_4(a) \\ c^{-1} x_5(a) \end{bmatrix} = 0 \right\}$$

and it satisfies for all $x \in D(\mathcal{A}_g)$

$$\langle \mathcal{A}_g\, x, x \rangle_{\mathcal{L}} = -\mu_w \left\| \partial_z (\rho_w^{-1} x_2) \right\|^2 - \xi \left\| (\rho_w^{-1} x_2 - \rho_u^{-1} x_4) \right\|^2 - k \left\| \partial_z (c^{-1} x_5) \right\|^2 .$$

Observe that in this case we have an equality in the equation above. This follows from equation (6.30) and because $W \Sigma W^T = 0$ since this gives $\langle \mathcal{A}_{\text{ext}}\, \tilde{x}, \tilde{x} \rangle_{\tilde{\mathcal{L}}} = 0$, see Theorem 6.23. $*$

6.6. Stability

In this section we show some results that can be useful to prove stability of the class of systems studied in this chapter. First note that some of the results presented in Chapter 4 and Section 5.2 also hold for first-order systems with dissipation since the no assumption is done on the matrix A_0 (or P_0 or G_0). The next result is only valid for systems with dissipation.

Theorem 6.33: *Consider the class of systems described in Theorem* 6.31*. If the operator $\mathcal{G}_R$ satisfies*

$$\langle \mathcal{G}_R^* \mathcal{L}x, S\mathcal{G}_R^* \mathcal{L}x \rangle \geq \varepsilon \|x\|_{\mathcal{L}}^2 + \frac{\kappa}{2} \begin{bmatrix} 0 & y^T \end{bmatrix} P_{W,\tilde{W}} \begin{bmatrix} 0 \\ y \end{bmatrix}, \quad \forall x \in D(\mathcal{A}_g), \tag{6.103}$$

for $\varepsilon > 0$ and $\kappa \leq 1$, then the system is exponentially stable.

PROOF: In order to prove exponential stability, we first let $u(t) = 0$ for $t \geq 0$. This implies that the state variable $x(t) \in D(\mathcal{A}_g)$. This together with equation (6.95) gives that the energy of the system satisfies

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \frac{1}{2} \begin{bmatrix} 0 & y^T(t) \end{bmatrix} P_{W,\tilde{W}} \begin{bmatrix} 0 \\ y(t) \end{bmatrix} - \langle \mathcal{G}_R^* \mathcal{L}x(t), S\mathcal{G}_R^* \mathcal{L}x(t) \rangle .$$

If inequality (6.103) holds then the equation above becomes

$$\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 \leq -2\varepsilon \|x(t)\|_{\mathcal{L}}^2 + (1-\kappa) \begin{bmatrix} 0 & y^T(t) \end{bmatrix} P_{W,\tilde{W}} \begin{bmatrix} 0 \\ y(t) \end{bmatrix}.$$

Now observe that $\begin{bmatrix} 0 & y^T(t) \end{bmatrix} P_{W,\tilde{W}} \left[\begin{smallmatrix} 0 \\ y(t) \end{smallmatrix}\right] \leq 0$, since this term corresponds to the expression $\langle \mathcal{A}_{\text{ext}} x, x \rangle$, see (6.30), and $\mathcal{A}_{\text{ext}}$ satisfies $\langle \mathcal{A}_{\text{ext}} x, x \rangle \leq 0$, see (6.28). Altogether gives

$$\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 \leq -2\varepsilon \, \|x(t)\|_{\mathcal{L}}^2 ,$$

provided that $\kappa \leq 1$. Thus,

$$\|x(t)\|_{\mathcal{L}}^2 \leq \mathrm{e}^{-2\varepsilon t} \|x(0)\|_{\mathcal{L}}^2 ,$$

which proves the exponential stability. ■

Example 6.34 Here we continue with Example 6.12 and we show that the fixed bed reactor with boundary conditions (6.43) is exponentially stable. We need to show that $\langle \mathcal{G}_R^* \mathcal{L}x, S\mathcal{G}_R^* \mathcal{L}x \rangle = \left\langle \frac{\partial x}{\partial z}, D\frac{\partial x}{\partial z} \right\rangle$ satisfies the condition in Theorem 6.33. By using the boundary conditions we obtain

$$\begin{aligned} \langle \mathcal{G}_R^* \mathcal{L}x, S\mathcal{G}_R^* \mathcal{L}x \rangle &= \left\langle \frac{\partial x}{\partial z}, D\frac{\partial x}{\partial z} \right\rangle = -\left\langle x, D\frac{\partial^2 x}{\partial z^2} \right\rangle - U|x(a)|^2 \\ &\geq \left\langle x, -D\frac{\partial^2 x}{\partial z^2} \right\rangle - \left(U|x(a)|^2 + U|x(b)|^2\right) \\ &= \left\langle x, -D\frac{\partial^2 x}{\partial z^2} \right\rangle - \left(\frac{1}{U}\|y\|_{\mathbb{R}}^2\right), \end{aligned} \tag{6.104}$$

where in the last step we used (6.44) together with the boundary conditions (6.43). The operator $-D\frac{\partial^2 x}{\partial z^2}$ (with domain $D(\mathcal{A}_g)$) is positive and it has the bounded inverse

$$\left(-D\frac{\partial^2}{\partial z^2}\right)^{-1} f(z) = -\left(\frac{1}{D}(z-a) + \frac{1}{U}\right)\int_a^b f(\tau)\,d\tau + \frac{1}{D}\int_a^z (z-\tau)f(\tau)\,d\tau.$$

This implies that $-D\frac{\partial^2 x}{\partial z^2}$ is coercive. Using this in the inequality (6.104) yields

$$\langle \mathcal{G}_R^* \mathcal{L}x, S\mathcal{G}_R^* \mathcal{L}x\rangle \geq \varepsilon \left\|x\right\|^2 - \frac{1}{U}\left\|y\right\|_{\mathbb{R}}^2 .$$

This is the inequality (6.103) with $\kappa = 1$, compare with (6.45). Therefore the system is exponentially stable by Theorem 6.33. $*$

Chapter 7

Power-Conserving Interconnection of Dirac Structures

In the previous chapters we studied the notion of Dirac structures and the relation with port-Hamiltonian systems (PHS). Depending on the selection of this Dirac structure we have defined energy preserving systems as well as dissipative systems. In this chapter we focus on the interconnection of these type of systems and we shall see the important role that the Dirac structure plays when doing that. In particular, we show that the interconnection of PHS is done by interconnecting the corresponding Dirac structures of the systems. The total PHS is then obtained from this new Dirac structure together with the sum of the Hamiltonians of the systems being interconnected. This property is useful when modeling systems using a modular approach where the system is thought of as the interconnection of smaller subsystems. This simplifies the modeling process since smaller subsystems are easier to model. This also allows to represent complex systems with components from different physical domains (e.g. mechanical, electrical, hydraulic) in a unified way. Furthermore, because of the modularity, the modeling process can be performed in an iterative manner, gradually refining the model by adding or changing other subsystems. Interconnection is also important from a control point of view, since implementing a control law or controlling a system is usually done by interconnecting a given system with an external device (controller) via the external variables (ports).

In this chapter we define what a power-conserving interconnection defined on the space of external variables is. We study the composition of the class of Dirac structures introduced in Chapters 2 and 6 and show that the power-conserving composition of these Dirac structures is again a Dirac structure and hence the interconnected systems is a PHS. Once we have the tools to study the composition of Dirac structures we can proceed to define BCS for the interconnected systems. We emphasize that the class of Dirac structures studied in this chapter are related to the skew-symmetric operators studied in Chapter 2. This chapter is mainly intended to give ideas on how to deal with interconnected systems and at the same time this ideas allow us to extend some results to a larger class of systems. Also, it shows how the general theory of the previous chapters can be applied by modeling in a modular way as port-Hamiltonian systems. We do not give results on general Dirac structures and the interconnection is restricted to be power-conserving. For more general cases we refer to [KvdSZ06] and [Gol02] and the references therein.

7.1. Port-variables and Dirac structures

As mentioned in the introduction, in this section we study the composition of a class of Dirac structures. Recall from Chapter 2 and 6 that the Dirac structure was fundamental in the definition of PHS. In General, a port-Hamiltonian system can be represented as in Figure 7.1. Central in the definition is the Dirac structure, denoted in Figure 7.1 by $\mathcal{D}_{\mathcal{J}}$. This Dirac structure links the various port-variables in such a way that the total power associated with the port-variables is zero.

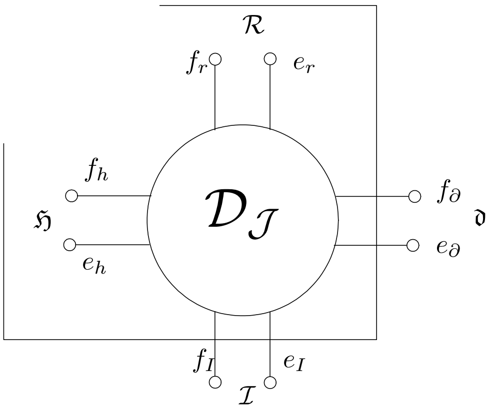

Figure 7.1.: Port-Hamiltonian system.

Note that the *port-variables* in Figure 7.1 have been split into different parts. First there are two *internal ports*: one, denoted by $\mathfrak{H}$, corresponds to energy storage and the other one, denoted by $\mathcal{R}$, corresponds to internal energy-dissipation. The other two ports, are the *external ports*: the one denoted by $\mathfrak{d}$ contains the boundary port-variables, and the other one, denoted by $\mathcal{I}$, corresponds to the distributed ports used to interconnect the system along the spatial domain. The external ports are used by the system to interact either with the environment or with other systems.

The port-variables associated with the energy storage port are denoted by $(f_h,\, e_h)$ (or simply $(f,\, e)$ when there might not be confusion with other ports, as it was done in Chapters 2 and 6). They are interconnected to the energy storage of the system which is defined by the state space X together with the Hamiltonian function $H : X \to \mathbb{R}$ denoting the energy. The interconnection of the energy storing elements to the energy storage port is done by setting (see Section 2.2)

$$f_h = \dot{x}, \qquad e_h = \frac{\partial H}{\partial x}(x).$$

The resistive port corresponds to energy dissipation (due to friction, resistance, heat transfer, etc.) and its port-variables are denoted by $(f_r,\, e_r)$. This is the type of ports used in Chapter 6 to deal with systems with dissipation. Finally, the port-variables associated with the boundary and distributed port will be denoted by $(f_\partial,\, e_\partial)$ and $(f_I,\, e_I)$, respectively. These are the variables that the system uses to interact with the environment or with other systems. Part of these ports could be used for control purposes. See [Pas06], [vdS05], [vdSM02], and [MvdSM04] and the references therein for more details on these type of ports and its usage.

In Chapter 2 we studied Dirac structures related to the skew-symmetric operator $\mathcal{J}$ given by (2.2). This clearly corresponds to a system where only energy storage and boundary ports are present. In Chapter 6 the resistive port was added and thus we considered the skew-symmetric operator $\mathcal{J}_e$ described in Section 6.1 as the operator related to the corresponding Dirac structure. In this chapter we add the distributed port and consider the interconnection of the resulting systems. In other words, we start by studying the Dirac structure related to the operator $\mathcal{J}_e$ described by

$$\mathcal{J}_e = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R & \mathcal{G}_I \\ -\mathcal{G}_R^* & 0 & 0 \\ -\mathcal{G}_I^* & 0 & 0 \end{pmatrix}, \qquad D(\mathcal{J}_e) = \begin{bmatrix} H^N(a,b,\mathbb{R}^n) \\ H^N(a,b,\mathbb{R}^{m_1}) \\ L_2(a,b)^{m_2} \end{bmatrix}, \tag{7.1}$$

where $\mathcal{J}$, $\mathcal{G}_R$, and $\mathcal{G}_I$ are given by given by

$$\begin{aligned} \mathcal{J}x &= \sum_{i=0}^{N} P_i \frac{\partial^i x}{\partial z^i}, \\ \mathcal{G}_R x &= \sum_{i=0}^{N} G_i \frac{\partial^i x}{\partial z^i}, \quad \mathcal{G}_R^* x = \sum_{i=0}^{N} (-1)^i G_i^T \frac{\partial^i x}{\partial z^i}, \\ \mathcal{G}_I &\in \mathcal{L}(L_2(a,b)^{m_2},\, L_2(a,b)^n), \end{aligned} \tag{7.2}$$

with $G_i \in \mathbb{R}^{n\times m_1}$, $P_i \in \mathbb{R}^{n\times n}$, $i = \{0, 1, \dots, N\}$, and $n > 0$, $m_1 \geq 0$, $m_2 \geq 0$. Furthermore, these matrices satisfy

$$P_i = (-1)^{i+1} P_i^T, \quad i = 0, 1, \dots, N. \tag{7.3}$$

Here, $\mathcal{G}_R^*$ is the formal adjoint of $\mathcal{G}_R$ and similarly for $\mathcal{G}_I^*$. Again, note that the assumption imposed on the matrices P_i means that $\mathcal{J}$ is formally skew symmetric.

Remark 7.1. Note that here we assume that $\mathcal{G}_I$ is a bounded operator. However, most of the results presented in this chapter also hold for differential operators of the same type as $\mathcal{G}_R$. We restrict $\mathcal{G}_I$ to be bounded for the sake of simplicity and clarity. Furthermore, this assumption is enough for most applications. ♣

It is now easy to see, from Section 6.2, that the following result holds.

Proposition 7.2: *The operator $\mathcal{J}_e$ defined by equations (7.1)–(7.3) is formally skew-symmetric and can be written as:*

$$\mathcal{J}_e \begin{bmatrix} e_h \\ e_r \\ e_I \end{bmatrix} = \begin{bmatrix} P_0 & G_0 & \mathcal{G}_I \\ -G_0^T & 0 & 0 \\ -\mathcal{G}_I^* & 0 & 0 \end{bmatrix} \begin{bmatrix} e_h \\ e_r \\ e_I \end{bmatrix} + \sum_{i=1}^{N} \underbrace{\begin{bmatrix} P_i & G_i \\ (-1)^{(i+1)} G_i^T & 0 \end{bmatrix}}_{\widetilde{P}_i} \frac{\partial^i}{\partial z^i} \begin{bmatrix} e_h \\ e_r \end{bmatrix} \tag{7.4}$$

where $\widetilde{P}_i \in \mathbb{R}^{(n+m_1)\times(n+m_1)}$ satisfies

$$\widetilde{P}_i = \begin{bmatrix} P_i & G_i \\ (-1)^{(i+1)} G_i^T & 0 \end{bmatrix} = (-1)^{i+1} \begin{bmatrix} P_i & G_i \\ (-1)^{i+1} G_i^T & 0 \end{bmatrix}^T = (-1)^{i+1} \widetilde{P}_i^T. \tag{7.5}$$

♡

Following the representation of $\mathcal{J}_e$ given by (7.4) it follows from Theorem 6.15 that the corresponding Stokes theorem applied to this $\mathcal{J}_e$ is

Theorem 7.3: *Consider the operator $\mathcal{J}_e$ defined by (7.1)–(7.3) and let r be the rank of Q given by (6.11) and M be an $(n+m_1)N \times r$ matrix whose columns span the range of Q. Then for any two functions e_1, $e_2 \in H^N(a,b)^{n+m_1+m_2}$ with $e_i = [e_{h,i},\ e_{r,i},\ e_{I,i}]^T$, $i = \{1, 2\}$, we have*

$$\int_a^b (\mathcal{J}_e e_1)^T(z)\, e_2(z)\, dz + \int_a^b e_1^T(z) (\mathcal{J}_e e_2)(z)\, dz$$
$$= \left\langle \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau\left(\begin{bmatrix} e_{h,1} \\ e_{r,1} \end{bmatrix} \right), \begin{bmatrix} \widetilde{Q} & 0 \\ 0 & -\widetilde{Q} \end{bmatrix} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau\left(\begin{bmatrix} e_{h,2} \\ e_{r,2} \end{bmatrix} \right) \right\rangle_{\mathbb{R}^{2r}}, \tag{7.6}$$

where $\widetilde{Q} = M^T Q M$, $M_Q = (M^T M)^{-1} M^T$, and $\tau(\cdot)$ is the operator described in Definition 2.5.

By the assumption on $\mathcal{G}_I$ and the theorem above it is easy to see that the boundary port-variables corresponding to the operator $\mathcal{J}_e$ in (7.1) are also described by Definition 6.17, i.e.,

$$\begin{bmatrix} \tilde{f}_{\partial,e} \\ \tilde{e}_{\partial,e} \end{bmatrix}_{|D(\mathcal{J}_e)} = \widetilde{R}_{\text{ext}} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau\left(\begin{bmatrix} e_h \\ e_r \end{bmatrix}\right), \quad e = [e_h^T,\, e_r^T,\, e_I^T]^T, \tag{7.7}$$

where the matrices $\widetilde{R}_{\text{ext}}$ and M_Q are described in Section 6.5. This follows since the matrix Q appearing in the Stokes theorem is the same that appears in Section 6.5. We also know, from Theorem 6.19, that in the case the matrix $\widetilde{P}_N$ in (7.5) is singular the boundary port operator $\left[\begin{smallmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{smallmatrix}\right]$ can be extended to the larger space described in (6.56), i.e.,

$$H(J_e,(a,b))^{n+m_1+m_2} = \left\{e \in L_2(a,b)^{n+m_1+m_2} \;\middle|\; \mathcal{J}_e\, e \in L_2(a,b)^{n+m_1+m_2}\right\}. \tag{7.8}$$

Observe that whenever $\widetilde{P}_N$ is nonsingular $H(J_e,(a,b))^{n+m_1+m_2}$ equals $D(\mathcal{J}_e)$ in (7.1). Next we define the flow and effort space as follows

$$\begin{gathered} \mathcal{F}^i = \mathcal{F}_h^i \times \mathcal{F}_r^i \times \mathcal{F}_I \times \mathcal{F}_\partial^i, \\ \mathcal{E}^i = \mathcal{E}_h^i \times \mathcal{E}_r^i \times \mathcal{E}_I \times \mathcal{E}_\partial^i, \\ \mathcal{F}_h^i = \mathcal{E}_h^i = L_2(a,b)^{n^i},\ \mathcal{F}_r^i = \mathcal{E}_r^i = L_2(a,b)^{m_1^i}, \\ \mathcal{F}_I = \mathcal{E}_I = L_2(a,b)^{m_2},\ \mathcal{F}_\partial^i = \mathcal{E}_\partial^i = \mathbb{R}^{r^i}. \end{gathered} \tag{7.9}$$

Thus, the bond space is given by $\mathcal{B} = \mathcal{F}^i \times \mathcal{E}^i$. By the selection of the port-variables, see (7.7), and Theorem 6.21 it is easy to see that the following result holds.

Theorem 7.4: *Consider the skew-symmetric operator $\mathcal{J}_e$ defined by* (7.1)–(7.3). *Let the boundary port-variables be described as in Definition* 6.17*, see* (7.7). *Then, the subspace $D_{\mathcal{J}_e}$ of $\mathcal{B}$ defined by*

$$\mathcal{D}_{\mathcal{J}_e} = \left\{ \begin{bmatrix} f \\ f_\partial \\ e \\ e_\partial \end{bmatrix} \in \mathcal{B} \;\middle|\; \begin{gathered} f = [f_h^T,\, f_r^T,\, f_I^T]^T,\ e = [e_h^T,\, e_r^T,\, e_I^T]^T, \\ e \in H(J_e,(a,b))^{n+m_1+m_2},\ \mathcal{J}_e\, e = f, \\ \begin{bmatrix} f_\partial, e \\ e_\partial, e \end{bmatrix}_{|D(\mathcal{J}_e)} = \widetilde{R}_{\text{ext}} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau\left(\begin{bmatrix} e_h \\ e_r \end{bmatrix}\right) \end{gathered} \right\}$$

is a Dirac structure with respect to (6.55)*, where $\widetilde{R}_{\text{ext}}$, M_Q, and $\tau(\cdot)$ are given according to Definition* 6.17.

7.2. Interconnection of Dirac structures

Recall from Chapter 6 that we interconnected the resistive port with a resistive relation in order to deal with systems with dissipation. In a similar manner we can also view the interconnection of the Dirac structures through the spatial domain, in which case we replace the resistive relation with another Dirac structure. Therefore, we consider two Dirac structures as described in Theorem 7.4; that is, let the Dirac structures $\mathcal{D}_i$, $i = \{1, 2\}$, be described by

$$\mathcal{D}_i = \left\{ \begin{bmatrix} f^i \\ f^i_\partial \\ e^i \\ e^i_\partial \end{bmatrix} \in \mathcal{B}^i \;\middle|\; \begin{array}{c} f^i = \begin{bmatrix} f^i_h \\ f^i_r \\ f^i_I \end{bmatrix}, \, e^i = \begin{bmatrix} e^i_h \\ e^i_r \\ e^i_I \end{bmatrix}, \\ e^i \in H(J^i_e, (a,b))^{n+m_1+m_2}, \; \mathcal{J}^i_e \, e^i = f^i, \\ \begin{bmatrix} f_{\partial, e^i} \\ e_{\partial, e^i} \end{bmatrix}_{|D(\mathcal{J}^i_e)} = \widetilde{R}^i_{\text{ext}} \begin{bmatrix} M^i_Q & 0 \\ 0 & M^i_Q \end{bmatrix} \tau\left(\begin{bmatrix} e^i_h \\ e^i_r \end{bmatrix} \right) \end{array} \right\}, \tag{7.10}$$

with

$$\mathcal{J}^i_e = \begin{pmatrix} \mathcal{J}_i & \mathcal{G}_{R_i} & \mathcal{G}_{I_i} \\ -\mathcal{G}^*_{R_i} & 0 & 0 \\ -\mathcal{G}^*_{I_i} & 0 & 0 \end{pmatrix}, \qquad D(\mathcal{J}^i_e) = \begin{bmatrix} H^N(a, b, \mathbb{R}^{n^i}) \\ H^N(a, b, \mathbb{R}^{m^i_1}) \\ H^N(a, b, \mathbb{R}^{m_2}) \end{bmatrix}. \tag{7.11}$$

As mentioned earlier the interconnection will take place through the spatial domain, that is, through the distributed port $\left[\begin{smallmatrix} f^i_I \\ e^i_I \end{smallmatrix}\right]$, see Figure 7.2.

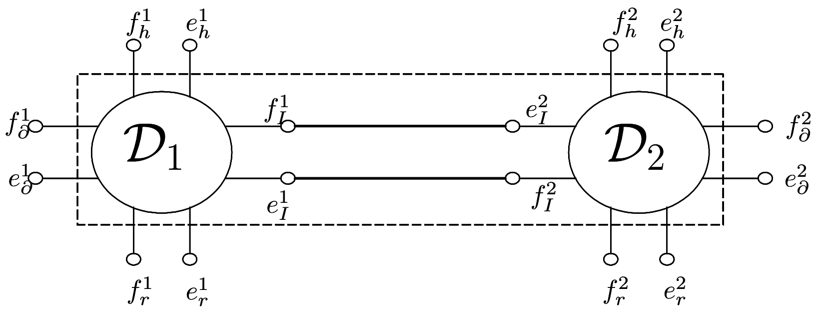

Figure 7.2.: Interconnection through the distributed port.

We want to interconnect the Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_2$ through the ports $\left[\begin{smallmatrix} f^i_I \\ e^i_I \end{smallmatrix}\right]$,

$i = \{1, 2\}$, see Figure 7.2, as follows

$$f_I^1 = \sigma e_I^2, \quad e_I^1 = -\sigma f_I^2, \quad \text{where } \sigma \text{ is either } 1 \text{ or } -1. \tag{7.12}$$

Observe that this is a power conserving interconnection since we have $f_I^1 e_I^1 + f_I^2 e_I^2 = 0$. We stress that in this chapter we only consider this type of power conserving interconnection, for a more thorough analysis of interconnection of Dirac structures see [KvdSZ06], [Pas06], and [Gol02] and the references therein.

Using (7.12) and the definition (7.10) restricted to $D(\mathcal{J}_e^i)$ with $\mathcal{J}_e^i$ given by (7.11) we obtain

$$\begin{aligned} f_I^1 &= -\mathcal{G}_{I_1}^* e_h^1 = \sigma e_I^2 \\ f_I^2 &= -\mathcal{G}_{I_2}^* e_h^2 = -\sigma e_I^1 \\ f_h^1 &= \mathcal{J}_1 e_h^1 + \mathcal{G}_{R_1} e_r^1 + \mathcal{G}_{I_1} e_I^1 = \mathcal{J}_1 e_h^1 + \mathcal{G}_{R_1} e_r^1 + \sigma \mathcal{G}_{I_1} \mathcal{G}_{I_2}^* e_h^2 \\ f_h^2 &= \mathcal{J}_2 e_h^2 + \mathcal{G}_{R_2} e_r^2 + \mathcal{G}_{I_2} e_I^2 = \mathcal{J}_2 e_h^2 + \mathcal{G}_{R_2} e_r^2 - \sigma \mathcal{G}_{I_2} \mathcal{G}_{I_1}^* e_h^1. \end{aligned}$$

From this and (7.10) (restricted to $D(\mathcal{J}_e^i)$) we see that the new interconnected structure $\mathcal{D} = \mathcal{D}_1 || \mathcal{D}_2$ is related to the new operator $\widetilde{\mathcal{J}}_e$ described by

$$\begin{bmatrix} f_h^1 \\ f_h^2 \\ f_r^1 \\ f_r^2 \end{bmatrix} = \underbrace{\begin{pmatrix} \mathcal{J}_1 & \sigma \mathcal{G}_{I_1} \mathcal{G}_{I_2}^* & \mathcal{G}_{R_1} & 0 \\ -\sigma \mathcal{G}_{I_2} \mathcal{G}_{I_1}^* & \mathcal{J}_2 & 0 & \mathcal{G}_{R_2} \\ -\mathcal{G}_{R_1}^* & 0 & 0 & 0 \\ 0 & -\mathcal{G}_{R_2}^* & 0 & 0 \end{pmatrix}}_{\widetilde{\mathcal{J}}_e} \begin{bmatrix} e_h^1 \\ e_h^2 \\ e_r^1 \\ e_r^2 \end{bmatrix}, \; D(\widetilde{\mathcal{J}}_e) = \begin{bmatrix} H^N(a,b)^{n^1} \\ H^N(a,b)^{n^2} \\ H^N(a,b)^{m_1^1} \\ H^N(a,b)^{m_1^2} \end{bmatrix} \tag{7.13}$$

By now it is easy to see that $\widetilde{\mathcal{J}}_e$ is a skew-symmetric operator (see e.g. Proposition 7.2) and by using the representation of $\mathcal{J}_i$ and G_{R_i} given in (7.2)–(7.3), i.e.,

$$\mathcal{J}_i x = \sum_{k=0}^{N} P_{i,k} \frac{\partial^k x}{\partial z^k}, \quad \mathcal{G}_{R_i} x = \sum_{k=0}^{N} G_{i,k} \frac{\partial^k x}{\partial z^k}, \quad i = \{1, 2\},$$

we can see that $\widetilde{\mathcal{J}}_e$ can be represented as

$$\begin{aligned} \widetilde{\mathcal{J}}_e \begin{bmatrix} e_h^1 \\ e_h^2 \\ e_r^1 \\ e_r^2 \end{bmatrix} &= \begin{bmatrix} P_{1,0} & \sigma \mathcal{G}_{I_1} \mathcal{G}_{I_2}^* & G_{1,0} & 0 \\ -\sigma \mathcal{G}_{I_2} \mathcal{G}_{I_1}^* & P_{2,0} & 0 & G_{2,0} \\ -G_{1,0}^T & 0 & 0 & 0 \\ 0 & -G_{2,0}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} e_h^1 \\ e_h^2 \\ e_r^1 \\ e_r^2 \end{bmatrix} \\ &+ \sum_{k=1}^{N} \underbrace{\begin{bmatrix} P_{1,k} & 0 & G_{1,k} & 0 \\ 0 & P_{2,k} & 0 & G_{2,k} \\ (-1)^{k+1} G_{1,k}^T & 0 & 0 & 0 \\ 0 & (-1)^{k+1} G_{2,k}^T & 0 & 0 \end{bmatrix}}_{\tilde{\Psi}_k} \frac{\partial^k}{\partial z^k} \begin{bmatrix} e_h^1 \\ e_h^2 \\ e_r^1 \\ e_r^2 \end{bmatrix}. \end{aligned} \tag{7.14}$$

Related to this operator we define

$$Q_e = \begin{pmatrix} \tilde{\Psi}_1 & \tilde{\Psi}_2 & \tilde{\Psi}_3 & \cdots & \tilde{\Psi}_{N-1} & \tilde{\Psi}_N \\ -\tilde{\Psi}_2 & -\tilde{\Psi}_3 & -\tilde{\Psi}_4 & \cdots & -\tilde{\Psi}_N & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ (-1)^{N-1}\tilde{\Psi}_N & 0 & \cdots & \cdots & & 0 \end{pmatrix}, \tag{7.15}$$

which is the Q matrix appearing when Stokes Theorem is applied to $\widetilde{\mathcal{J}}_e$, see e.g. Theorem 7.3. In fact, Stokes theorem applied to this operator gives

Theorem 7.5: *Consider the skew-symmetric operator* $\widetilde{\mathcal{J}}_e$ *defined by* (7.13) *with domain* $D(\widetilde{\mathcal{J}}_e) = H^N(a,b)^{n^1+n^2+m_1^1+m_1^2}$, *see* (7.11). *Let* r *be the rank of* Q_e *given by* (7.15) *and* M *be an* $(n^1+n^2+m_1^1+m_1^2)N \times r$ *matrix whose columns span the range of* Q_e. *Then for any two functions* $e_1, e_2 \in D(\widetilde{\mathcal{J}}_e)$ *with* $e_i = \left[(e_h^1)^T, (e_h^2)^T, (e_r^1)^T, (e_r^2)^T\right]^T$, $i = \{1,2\}$, *we have*

$$\begin{aligned} &\int_a^b (\widetilde{\mathcal{J}}_e e_1)^T(z)\, e_2(z)\, dz + \int_a^b e_1^T(z)(\widetilde{\mathcal{J}}_e e_2)(z)\, dz \\ &= \left(\begin{bmatrix} e_1^T(z), \ldots & , \frac{d^{N-1}}{dz^{N-1}} e_1^T(z) \end{bmatrix} M_Q^T\, \widetilde{Q}\, M_Q \begin{bmatrix} e_2(z) \\ \vdots \\ \frac{d^{N-1}}{dz^{N-1}} e_2(z) \end{bmatrix} \right)_a^b \\ &= \left\langle \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau(e_1), \begin{bmatrix} \widetilde{Q} & 0 \\ 0 & -\widetilde{Q} \end{bmatrix} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau(e_2) \right\rangle_{\mathbb{R}^{2r}}, \end{aligned} \tag{7.16}$$

where $\widetilde{Q} = M^T Q_e M$, $M_Q = (M^T M)^{-1} M^T$, *and* $\tau(\cdot)$ *is the operator described in Definition* 2.5.

From this it is easy to see that the corresponding boundary port-variables, which can be extended to a larger space, are described by Definition 6.17, i.e., (with the notation of Theorem 7.5)

$$\begin{bmatrix} \tilde{f}_{\partial,e} \\ \tilde{e}_{\partial,e} \end{bmatrix}_{|D(\widetilde{\mathcal{J}}_e)} = \widetilde{R}_{\text{ext}} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau(e), \quad e = \left[(e_h^1)^T, (e_h^2)^T, (e_r^1)^T, (e_r^2)^T\right]^T, \tag{7.17}$$

where the matrix $\widetilde{R}_{\text{ext}}$ is described in Section 6.5 and is given by

$$\tilde{R}_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{Q} & -\tilde{Q} \\ I & I \end{bmatrix}. \tag{7.18}$$

It is worth noticing that these port-variables are just a rearrangement of the port-variables corresponding to the Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_2$. This follows from the following lemma.

Lemma 7.6: *Consider the port-variables of the Dirac structures* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *described by* (7.10) *and the port-variables corresponding to the operator* $\widetilde{\mathcal{J}}_e$ *given by* (7.17)–(7.18). *Then, there exist a unitary matrix* U *such that*

$$U^T\, Q_e\, U = \left[\begin{array}{c|c} Q^1 & 0 \\ \hline 0 & Q^2 \end{array}\right]$$

and for $e = \left[(e_h^1)^T,\, (e_h^2)^T,\, (e_r^1)^T,\, (e_r^2)^T\right]^T$

$$U^T \begin{bmatrix} e(z) \\ \vdots \\ \frac{d^{N-1}}{dz^{N-1}} e(z) \end{bmatrix} = \begin{bmatrix} e_h^1 \\ e_r^1 \\ \vdots \\ \frac{d^{N-1}}{dz^{N-1}} e_h^1 \\ \frac{d^{N-1}}{dz^{N-1}} e_r^1 \\ e_h^2 \\ e_r^2 \\ \vdots \\ \frac{d^{N-1}}{dz^{N-1}} e_h^2 \\ \frac{d^{N-1}}{dz^{N-1}} e_r^2 \end{bmatrix}. \qquad \heartsuit$$

PROOF: Just let U be given by

$$U = \left[\begin{array}{ccccccccccc|ccccccc}
I & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & & I & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & & 0 & I & \dots & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & & & \vdots & & & & & & \vdots & & & \\
\hline
0 & 0 & 0 & 0 & \dots & I & 0 & \dots & 0 & 0 & & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & & 0 & 0 & \dots & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & I & \dots & 0 & 0 & & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & & 0 & 0 & \dots & 0 & I & 0 & 0 \\
 & & \vdots & & & & & & \vdots & & & & & & \vdots & & & \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & I & & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & & 0 & 0 & \dots & 0 & 0 & 0 & I
\end{array}\right],$$

where each entry multiplies an entry in the matrix $\tilde{\Psi}_k$ appearing in (7.14) and equation (7.15). For instance, in the first order case, i.e., $N = 1$, we have

$$U^T Q_e U = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_{1,1} & 0 & G_{1,1} & 0 \\ 0 & P_{2,1} & 0 & G_{2,1} \\ G_{1,1}^T & 0 & 0 & 0 \\ 0 & G_{2,1}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} P_{1,1} & G_{1,1} & 0 & 0 \\ G_{1,1}^T & 0 & 0 & 0 \\ 0 & 0 & P_{2,1} & G_{2,1} \\ 0 & 0 & G_{2,1} & 0 \end{bmatrix} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}. \qquad \blacksquare$$

Now we can prove that the interconnected structure is indeed a Dirac structure. Since we know that $\widetilde{\mathcal{J}}_e$ is a skew symmetric operator, then we can apply the results of Section 6.5. In particular, from Theorem 6.21, we have the following results.

Theorem 7.7: *Consider the skew-symmetric operator $\widetilde{\mathcal{J}}_e$ defined by* (7.13). *Let the boundary port-variables be described as above, see* (7.17), *and define the bond space as $\widetilde{\mathcal{B}} = \widetilde{\mathcal{F}} \times \widetilde{\mathcal{E}}$ with $\widetilde{\mathcal{F}} = \widetilde{\mathcal{E}} = \mathcal{F}_h^1 \times \mathcal{F}_h^2 \times \mathcal{F}_r^1 \times \mathcal{F}_r^2 \times \mathcal{F}_\partial^1 \times \mathcal{F}_\partial^2$, see* (7.9). *Then, the subspace $\mathcal{D}_{\mathcal{J}_e}$ of $\widetilde{\mathcal{B}}$ defined by*

$$\mathcal{D}_{\mathcal{J}_e} = \left\{ \begin{bmatrix} f \\ f_\partial \\ e \\ e_\partial \end{bmatrix} \in \widetilde{\mathcal{B}} \;\middle|\; \begin{array}{c} f = \left[(f_h^1)^T, \, (f_h^2)^T, \, (f_r^1)^T, \, (f_r^2)^T \right]^T, \\ e = \left[(e_h^1)^T, \, (e_h^2)^T, \, (e_r^1)^T, \, (e_r^2)^T \right]^T, \\ e \in H(\widetilde{\mathcal{J}}_e, (a,b))^{(n^1+n^2+m_1^1+m_1^2)}, \; \widetilde{\mathcal{J}}_e \, e = f, \\ \begin{bmatrix} \tilde{f}_\partial, e \\ \tilde{e}_\partial, e \end{bmatrix}_{|D(\widetilde{\mathcal{J}}_e)} = \widetilde{R}_{\text{ext}} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix} \tau(e) \end{array} \right\}$$

is a Dirac structure with respect to (2.14), *where $\widetilde{R}_{\text{ext}}$, M_Q, and $\tau(\cdot)$ are given according to Definition* 6.17.

7.3. Boundary control systems: Examples

Now that we have that the power-conserving interconnection of Dirac structures is again a Dirac structure, it is easy to parameterize boundary control systems out of this interconnected structures, even if the system has dissipation. In fact, the results presented in Section 6.5 hold for these interconnected systems, since the only requirement is that the differential operator related to the Dirac structure is skew-symmetric. The following examples may help to clarify the ideas. For more details see Section 6.5.

Example 7.8 (Suspension system) Consider a suspension system described by two strings (or two beams for more accurate model) connected in parallel through a distributed spring. This system can be described as the interconnection of three subsystems, i.e., two vibrating strings and one distributed spring. For the sake of completeness we follow the modeling process of the strings so that the physical meaning of the distributed port-variables is clear.

Model of the strings: Consider a flexible string held stationary at both ends and free to vibrate transversely subject to the restoring forces due to tension in the string and to an external force $F(z,t)$. Our hypothesis is that the string is under constant tension, T, established when it was stretched between its fixed end points. The transverse displacement at position z along the string at time t is denoted $u(z,t)$.

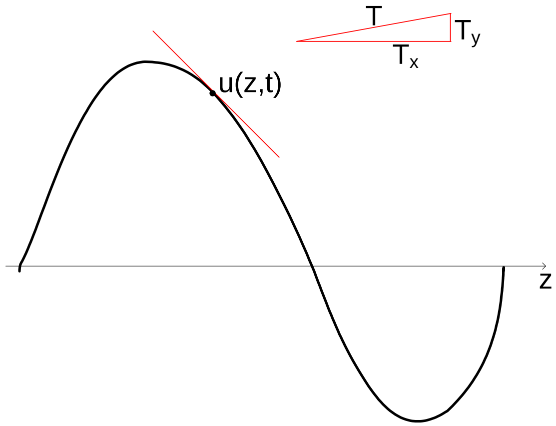

Figure 7.3.: Vibrating string.

The tangent to the string at $u(z,t)$ is indicated in the figure. Also shown is triangle with legs T_x (the component of tension along the x-direction), T_y (that parallel to displacement u) and hypotenuse T (tension in the string). If we suppose that displacements are small then T_x is approximately equal to T; but T_y is given by

$$T_y = T\frac{\partial u}{\partial z}. \tag{7.19}$$

Define the strain

$$q(z,t) = \frac{\partial u}{\partial z}. \tag{7.20}$$

Using this, equation (7.19) becomes

$$T_y = Tq(z,t). \tag{7.21}$$

To mathematically describe the vibrations on the string, we consider all the forces acting on a small section of the string as shown in Figure 7.4.

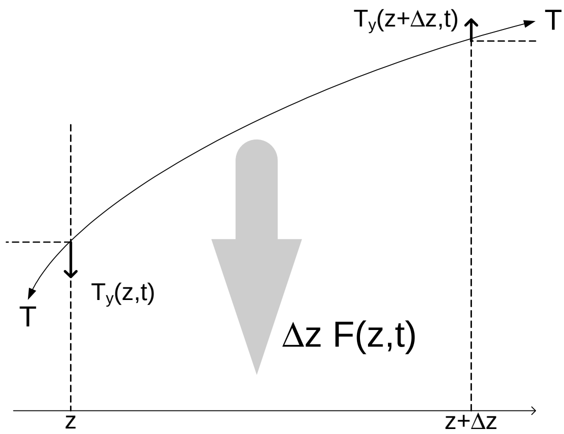

Figure 7.4.: Small segment of the vibrating string.

Essentially, the wave equation is nothing more than Newton's equation of motion applied to the string (the change in momentum mu_{tt} of a small string segment is equal to the applied forces). Looking at Figure 7.4 we can see that the forces acting on the string in the direction perpendicular to the z-axis are $T_y(z+\Delta z,t)$, $T_y(z,t)]$ and $\Delta z F(z,t)$. If we now apply Newton's equation of motion to the small segment of the string and use (7.21), we obtain

$$\begin{aligned} \rho\Delta z \frac{\partial^2 u}{\partial t^2}(z,t) &= [T_y(z+\Delta z,t) - T_y(z,t)] - \Delta z F(z,t) \\ \frac{\partial p}{\partial t}(z,t) &= T\frac{1}{\Delta z}[q(z+\Delta z,t) - q(z,t)] - F(z,t) \end{aligned} \tag{7.22}$$

where

$$p(z,t) = \rho\frac{\partial u}{\partial t}(z,t), \tag{7.23}$$

ρ is the mass density, $m = \rho\Delta z$ is the mass, and $\rho\Delta z p(z,t)$ is the momentum distribution. By letting $\Delta z \to 0$ in equation (7.22) yields

$$\frac{\partial p}{\partial t} = T\frac{\partial q}{\partial z}(z,t) - F(x,t). \tag{7.24}$$

Observe that from equations (7.20) and (7.23) we obtain

$$\frac{\partial q}{\partial t} = \frac{\partial^2 u}{\partial z \partial t} = \frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{7.25}$$

Furthermore, the energy of the system is given by

$$\begin{aligned}\mathcal{H}(t) =& \frac{1}{2}\int_a^b \left(\rho|u_t|^2 + T|u_x|^2\right)\, dz \\ =& \frac{1}{2}\int_a^b \left(\frac{1}{\rho}|p|^2 + T|q|^2\right)\, dz.\end{aligned} \tag{7.26}$$

Hence, the state variables are $x = \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ and the effort variables are given by

$$e = \begin{bmatrix} \partial_p H \\ \partial_q H \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho}p \\ Tq \end{bmatrix}, \tag{7.27}$$

where H is the energy density. Thus, from the equation above, (7.24) and (7.25) we can see that the vibrating string can be described by

$$\frac{\partial}{\partial t}\underbrace{\begin{bmatrix} p(z,t) \\ q(z,t) \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial z}}_{\mathcal{J}}\begin{bmatrix} \frac{1}{\rho}p \\ Tq \end{bmatrix} + \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{\mathcal{G}_I} F(z,t). \tag{7.28}$$

Distributed spring: The spring can be described in port-Hamiltonian form by

$$\begin{aligned}\frac{\partial q}{\partial t} =& e_{s_1} - e_{s_2} = [\underbrace{1}_{\tilde{G}_{I_1}}, \underbrace{-1}_{\tilde{G}_{I_2}}]\begin{bmatrix} e_{s_1} \\ e_{s_2} \end{bmatrix} \\ f_{s_1} =& -e_s \\ f_{s_1} =& e_s,\end{aligned} \tag{7.29}$$

where $q = z_1 - z_2$ is the elongation of the spring, e_{s_1} and e_{s_2} are the velocities, $e_s = kq$ is the force, k is the constant of the spring and f_{s_1}, f_{s_2} are the forces applied to the spring. The ports used to interact with the environment are the forces f_{s_1} and f_{s_2} as well as the velocities e_{s_1} and e_{s_2}. Note that in this case we have two ports and thus two corresponding operators $\tilde{G}_{I_i}$, $i = \{1, 2\}$. The energy of the system is

$$H_s = \frac{1}{2}\int_a^b kq^2\, dz. \tag{7.30}$$

Observe that in this system there is no skew-symmetric operator, i.e., $\mathcal{J}_s = 0$, since this operator represents the canonical coupling between two physical domains: the kinetic and the potential (internal) domain, and in this case we only have potential energy.

Interconnected system: In summary, the vibrating strings can be described in port-Hamiltonian form as

$$\begin{aligned}\frac{\partial}{\partial t}\begin{bmatrix} p_i \\ q_i \end{bmatrix} =& \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial z}\begin{bmatrix} \frac{1}{\rho_i}p_i \\ T_i q_i \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e_{I_i} = \mathcal{J}_i e_{h_i} + \mathcal{G}_{I_i} e_{I_i} \\ f_{I_i} =& -\mathcal{G}_{I_i}^T e_{h_i} = \frac{1}{\rho_i}p_i,\end{aligned} \tag{7.31}$$

where

$$\mathcal{G}_{I_i} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad e_{h_i} = \begin{bmatrix} \frac{1}{\rho_i} p_i \\ T_i q_i \end{bmatrix} = \begin{bmatrix} \text{velocity} \\ \text{stress} \end{bmatrix}, \quad p_i = \rho_i \frac{\partial u_i}{\partial t}, \quad q_i = \frac{\partial u_i}{\partial x}, \tag{7.32}$$

and e_{I_i} is a force acting along the i-th string, $i = \{1, 2\}$. Recall, that we also need to define boundary conditions. Observe that the ports used to interact with the environment are the force e_{I_i} and the velocity f_{I_i}, as well as the boundary ports.

Next, we consider the interconnection of the two strings and the spring. Here we connect the forces e_{I_1}, e_{I_2}, with f_{s_1}, f_{s_2}, as well as the velocities f_{I_1}, f_{I_2} with e_{s_1}, e_{s_2}. The connection is as follows (see Figure 7.5)

$$f_{I_1} = -e_{s_1},\, e_{I_1} = f_{s_1}, \quad f_{I_2} = -e_{s_2},\, e_{I_2} = f_{s_2}. \tag{7.33}$$

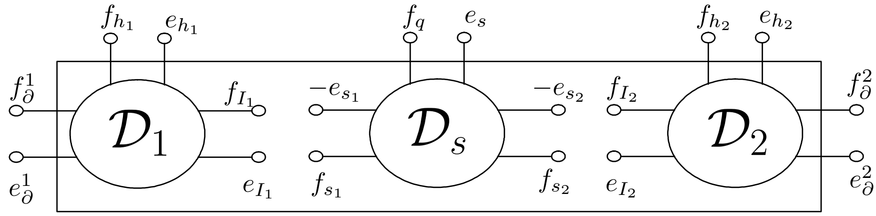

Figure 7.5.: Interconnection of the system.

This interconnection corresponds to $\sigma = -1$, see (7.12), and thus we have a power-conserving interconnection. It thus follows that the interconnected system is described by the operator $\widetilde{\mathcal{J}}_e$ given by (7.13) or (7.14) (with $\mathcal{G}_R$ an empty operator), i.e.,

$$\begin{bmatrix} f_{h_1} \\ f_q \\ f_{h_2} \end{bmatrix} = \begin{bmatrix} \mathcal{J}_1 & \sigma\,\mathcal{G}_{I_1}\tilde{G}_{I_1}^* & 0 \\ -\sigma\,\tilde{G}_{I_1}\mathcal{G}_{I_1}^* & \mathcal{J}_s & -\sigma\,\tilde{G}_{I_2}\mathcal{G}_{I_2}^* \\ 0 & \sigma\,\mathcal{G}_{I_2}\tilde{G}_{I_2}^* & \mathcal{J}_2 \end{bmatrix} \begin{bmatrix} e_{h_1} \\ e_s \\ e_{h_2} \end{bmatrix},$$

where $\mathcal{J}_s = 0$. Thus the dynamics of the system can be described (by using (7.29)–(7.32) on the equation above) with $e_{h_i} = \left[\begin{smallmatrix} \rho_i^{-1} p_i \\ T_i q_i \end{smallmatrix}\right]$, $i = \{1, 2\}$, and $e_s = k\,q$ by

$$\frac{\partial}{\partial t}\left[\begin{array}{c} p_1 \\ q_1 \\ \hline q \\ \hline p_2 \\ q_2 \end{array}\right] = \left[\begin{array}{cc|c|cc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \frac{\partial}{\partial z} \begin{bmatrix} e_{h_1} \\ e_s \\ e_{h_2} \end{bmatrix} + \left[\begin{array}{cc|c|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} e_{h_1} \\ e_s \\ e_{h_2} \end{bmatrix}$$

together with the port-variables

$$\begin{bmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{bmatrix} = \begin{bmatrix} (T_1 q_1)(b) - (T_1 q_1)(a) \\ (\rho_1^{-1} p_1)(b) - (\rho_1^{-1} p_1)(a) \\ (T_2 q_2)(b) - (T_2 q_2)(a) \\ (\rho_2^{-1} p_2)(b) - (\rho_2^{-1} p_2)(a) \\ (T_1 q_1)(b) + (T_1 q_1)(a) \\ (\rho_1^{-1} p_1)(b) + (\rho_1^{-1} p_1)(a) \\ (T_2 q_2)(b) + (T_2 q_2)(a) \\ (\rho_2^{-1} p_2)(b) + (\rho_2^{-1} p_2)(a) \end{bmatrix}.$$

The total energy $\mathcal{H}$ is given by the sum of the three elements in the interconnection, i.e.,

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_s = \frac{1}{2} \int_a^b \left(\frac{1}{\rho_1} |p_1|^2 + T_1 |q_1|^2 + \frac{1}{\rho_2} |p_2|^2 + T_2 |q_2|^2 + k q^2 \right) dz. \tag{7.34}$$

It thus follows that we can use some of the results of the previous chapters again to this interconnected system. *

It is also possible to interconnect finite-dimensional systems with the boundary of ($1D$) infinite-dimensional systems. In these cases it may be better (not always) to define the input and output of the infinite-dimensional system rather than working directly with the boundary port-variables. This was the case of Section 5.1.2 where first the input and output of the system were selected and the closed-loop system was studied. The example below shows the main ideas.

Example 7.9 (Transmission line, RLC circuit) As an example consider a transmission line interconnected to two electrical circuits through the boundary. For simplicity we consider an RLC circuit connected at $z = b$ and a controller connected at $z = a$, see Figure 7.6.

RLC circuit: Consider an RLC circuit whose port-Hamiltonian model is given by

$$\frac{d}{dt} \begin{bmatrix} q \\ \phi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R_b \end{bmatrix} \begin{bmatrix} \frac{1}{C_b} q \\ \frac{1}{L_b} \phi \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_b \tag{7.35}$$

$$\iota_b = [0,\ 1] \begin{bmatrix} \frac{1}{C_b} q \\ \frac{1}{L_b} \phi \end{bmatrix} = \frac{1}{L_b} \phi,$$

where q is the charge stored in the capacitor, ϕ the flux in the inductance, $\frac{1}{C_b} q$ is the voltage, and $\frac{1}{L_b} \phi$ is the current. Note that the ports corresponds to the voltage applied to the circuit, v_b, and the current in the circuit, ι_b. The energy function is

$$E_b(t) = \frac{1}{2} \left(\frac{1}{C_b} q(t)^2 + \frac{1}{L_b} \phi(t)^2 \right). \tag{7.36}$$

Transmission line: The model of the transmission line is

$$\frac{\partial}{\partial t}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\frac{\partial}{\partial z}}_{\mathcal{J}}\begin{bmatrix} \frac{1}{C}\,x_1 \\ \frac{1}{L}\,x_2 \end{bmatrix}, \tag{7.37}$$

where $x_1(z,t)$ is the charge density, $\frac{1}{C}\,x_1(z,t)$ is the distributed voltage , $x_2(z,t)$ the flux density, and $\frac{1}{L}\,x_2(z,t)$ is the distributed current. The energy function is

$$E(t) = \frac{1}{2}\int_a^b \left(\frac{1}{C}|x_1|^2 + \frac{1}{L}|x_2|^2\right)\,dz. \tag{7.38}$$

Let $e = \left[\begin{smallmatrix} e_q \\ e_\phi \end{smallmatrix}\right] = \left[\begin{smallmatrix} \frac{1}{C}\,x_1 \\ \frac{1}{L}\,x_2 \end{smallmatrix}\right] = \mathcal{L}\,x$. In this case the boundary port variables are

$$\begin{bmatrix} f_{\partial,x} \\ e_{\partial,x} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}\begin{bmatrix} e(b) \\ e(a) \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} -e_\phi(b)+e_\phi(a) \\ -e_q(b)+e_q(a) \\ e_q(b)+e_q(a) \\ e_\phi(b)+e_\phi(a) \end{bmatrix} = \begin{bmatrix} f_{\partial_1} \\ f_{\partial_2} \\ e_{\partial_1} \\ e_{\partial_2} \end{bmatrix}. \tag{7.39}$$

As input we select the voltage at both ends, i.e.,

$$u = \frac{1}{L}\begin{bmatrix} x_2(a) \\ -\,x_2(b) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \Rightarrow \quad W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{7.40}$$

Thus, as outputs we select the currents at the boundary, i.e.,

$$y = \mathcal{C}(\mathcal{L}\,x) = \frac{1}{C}\begin{bmatrix} x_1(a) \\ x_1(b) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \Rightarrow \quad \widetilde{W} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}. \tag{7.41}$$

It thus follows that the system is an impedance energy preserving system, i.e., the energy satisfies $\frac{d}{dt}E(t) = u(t)^T y(t)$.

Controller: For simplicity, we assume that the controller can be represented by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -R_a \end{bmatrix}\begin{bmatrix} \kappa_1\,\xi_1 \\ \kappa_2\,\xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_a \\ \iota_a &= [0,\ 1]\begin{bmatrix} \kappa_1\,\xi_1 \\ \kappa_2\,\xi_2 \end{bmatrix} = \kappa_2\,\xi_2, \end{aligned} \tag{7.42}$$

where κ_1 and κ_2 are positive constants. Furthermore the energy function is

$$E_a(t) = \frac{1}{2}\left(\frac{1}{C}\,q(t)^2 + \frac{1}{L}\,\phi(t)^2\right). \tag{7.43}$$

Interconnection: Now we can proceed to interconnect the three subsystems. The interconnection is as follows

$$-u_1 = \iota_a,\ y_1 = v_a, \quad \text{and} \quad -\,u_2 = \iota_b,\ y_2 = v_b. \tag{7.44}$$

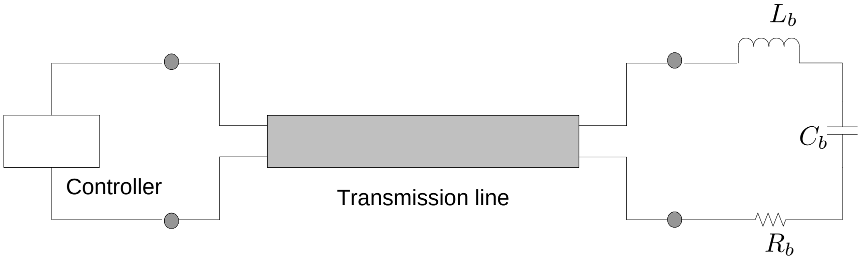

Figure 7.6.: Interconnected transmission line.

Thus the dynamics of the interconnected system are described by

$$\frac{\partial}{\partial t}\begin{bmatrix} x \\ \vartheta \end{bmatrix} = \begin{bmatrix} \mathcal{J} & 0 \\ B_\alpha \mathcal{C} & A_\alpha \end{bmatrix}\begin{bmatrix} \mathcal{L}\, x \\ \mathcal{L}_\vartheta\, \vartheta \end{bmatrix},$$
$$u = W\begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} = -C_\alpha \mathcal{L}_\vartheta\, \vartheta \tag{7.45}$$

where $\mathcal{L} = \operatorname{diag}\{C^{-1},\, L^{-1}\}$, $\mathcal{L}_\vartheta = \operatorname{diag}\{\kappa_1,\, \kappa_2, C_b^{-1},\, L_b^{-1}\}$, $\mathcal{J}$ is described in equation (7.37), $\mathcal{C}\,(\mathcal{L}\, x) = y$ (see (7.41)), $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$, and

$$\vartheta = \left[\begin{array}{c} \xi_1 \\ \xi_2 \\ \hline q \\ \phi \end{array}\right],\, A_\alpha = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & -R_a & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -R_b \end{array}\right),\, B_\alpha = \left(\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right) = C_\alpha^T.$$

Existence of solutions: To check the existence of solutions we can use similar ideas to those presented in Section 5.1.2. In general we consider systems of the form presented in equation (7.45) where $\mathcal{J}$ is a skew-symmetric differential operator as described at the beginning of Chapter 2, $\mathcal{L}$ and $\mathcal{L}_\vartheta$ are coercive operators and $A_\alpha \in \mathbb{R}^{\eta_1\times\eta_1}$, $B_\alpha \in \mathbb{R}^{\eta_1\times\eta_2}$ and $C_\alpha \in \mathbb{R}^{\eta_2\times\eta_1}$, see Remark 5.7. Furthermore, there exist matrices $P = P^T > 0$ and Q such that

$$P\, A_\alpha + A_\alpha^T\, P = -Q^T Q, \qquad P\, B_\alpha = C_\alpha^T. \tag{7.46}$$

As state space we define $\widetilde{X} = X \oplus \mathbb{R}^{\eta_1}$ where X is described in (2.33). The inner product on $\widetilde{X}$ is defined for $w_i = \left[\begin{smallmatrix} x_i \\ \vartheta_i \end{smallmatrix}\right] \in \widetilde{X}$, $i = \{1, 2\}$, as

$$\langle w_1, w_2\rangle_{\widetilde{X}} = \langle x_1, x_2\rangle_{\mathcal{L}} + \frac{1}{2}\,\langle \mathcal{L}_\vartheta\, P\, \vartheta_1, \vartheta_2\rangle_{\mathbb{R}} + \frac{1}{2}\,\langle P\, \mathcal{L}_\vartheta\, \vartheta_2, \vartheta_1\rangle_{\mathbb{R}}\,, \tag{7.47}$$

where P is the positive definite matrix found in (7.46).

Theorem 7.10: *Consider the interconnected system* (7.45) *as described above where the open-loop distributed parameter system is an impedance energy preserving system, i.e.,* $\frac{d}{dt}E(t) = u(t)^T y(t)$ *holds and the finite-dimensional system satisfies* (7.46)*. Then, the interconnected system is a boundary control system on* $\widetilde{X}$*. Furthermore, the operator* $\mathcal{A}_c$ *defined by*

$$\mathcal{A}_c\, w = \begin{bmatrix} \mathcal{J} & 0 \\ B_\alpha \mathcal{C} & A_\alpha \end{bmatrix} \begin{bmatrix} \mathcal{L}\, x \\ \mathcal{L}_\vartheta\, \vartheta \end{bmatrix} \tag{7.48a}$$

with domain

$$\left\{ w = \begin{bmatrix} x \\ v \end{bmatrix} \in \begin{bmatrix} X \\ \mathbb{R}^m \end{bmatrix} \,\middle|\, \mathcal{L}\, x \in H^N(a,b;\mathbb{R}^n),\ \text{and}\ \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \\ \mathcal{L}_\vartheta\, \vartheta \end{bmatrix} \in \ker \widetilde{W}_D \right\}, \tag{7.48b}$$

where

$$\widetilde{W}_D = \begin{bmatrix} W & C_\alpha \end{bmatrix}, \tag{7.48c}$$

generates a contraction semigroup.

PROOF: The proof follows the same ideas as those presented in the proof of Theorem 5.8. First we need to prove the existence of the operator $\mathfrak{R} \in \mathcal{L}(U, \widetilde{X})$. This follows since the matrix W is full-rank (see the first paragraph in the proof of Theorem 5.8).

Next we need to prove that $\mathcal{A}_c$ generates a semigroup. We will use the Lümer-Phillips theorem (see [Paz83]). First we prove that $\langle \mathcal{A}_c w, w\rangle \le 0$. Let $w = \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(\mathcal{A}_c)$, then using (7.47) we have (recall that X is a real Hilbert space)

$$\begin{aligned}
\langle \mathcal{A}_c w, w\rangle_{\tilde{X}} &= \langle \mathcal{J}\mathcal{L}x, x\rangle_{\mathcal{L}} + \frac{1}{2}\langle PA_\alpha \mathcal{L}_\vartheta\, \vartheta + PB_\alpha y, \mathcal{L}_\vartheta \vartheta\rangle_{\mathbb{R}} \\
&\quad + \frac{1}{2}\langle \mathcal{L}_\vartheta \vartheta, PA_\alpha \mathcal{L}_\vartheta\, \vartheta + PB_\alpha y\rangle_{\mathbb{R}} \\
&= \left\langle \frac{\partial x}{\partial t}, x\right\rangle_{\mathcal{L}} + \frac{1}{2}\left\langle (PA_\alpha + A_\alpha^T P)\mathcal{L}_\vartheta\, \vartheta, \mathcal{L}_\vartheta \vartheta\right\rangle_{\mathbb{R}} + \langle \mathcal{L}_\vartheta \vartheta, PB_\alpha y\rangle_{\mathbb{R}} \\
&= u^T y + \frac{1}{2}\left\langle -Q^T Q\, P\mathcal{L}_\vartheta\, \vartheta, \mathcal{L}_\vartheta \vartheta\right\rangle_{\mathbb{R}} + \langle C_\alpha\, \mathcal{L}_\vartheta \vartheta, y\rangle_{\mathbb{R}},
\end{aligned}$$

where we used (7.46). Using the lower equation in (7.45) we obtain

$$\langle \mathcal{A}_c w, w\rangle_{\tilde{X}} = -\frac{1}{2}\left\langle Q^T Q\, P\mathcal{L}_\vartheta\, \vartheta, \mathcal{L}_\vartheta \vartheta\right\rangle_{\mathbb{R}} \le 0.$$

Now we need to prove that $(I - \mathcal{A}_c)$ is equal to $\widetilde{X}$. This follows as the last part of the proof of Theorem 5.8, see page 122. Hence the result follows from the Lümer-Phillips theorem. ∎

Continuing with the transmission line example, we clearly have that $P = I$ and

$$-Q^T Q = -2 \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & R_a & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_b \end{array} \right).$$

It thus follows that this is a boundary control system. $*$

Chapter 8

$2\mathrm{D}$ and $3\mathrm{D}$ Boundary Control Systems

In the previous chapters we have concentrated in one dimensional ($1D$) systems. In Chapter 2 we parameterized a class of BCS based on a selection of a matrix, which determines the boundary conditions. In this chapter we present some ideas that could help to extend those results to systems in larger spatial domains. We introduce what could be the starting point for that extension. In particular, we concentrate on $2D$ and $3D$ systems. Obviously, there are some differences and some complications emerge when we study $2D$ and $3D$ systems as compared to $1D$ systems. Mainly, the input an output spaces are no longer finite-dimensional. This in turn, implies that the matrix W that determines the boundary conditions is not any more a matrix, but an operator depending on the position on the boundary of the spatial domain. Also, the results obtained depend on the shape (smoothness) of the boundary of the spatial domain. The main complication emerges with the boundary operator and the selection of the input and output spaces since these selection may depend on the problem at hand.

8.1. Basic concepts on Sobolev spaces

We start with a basic introduction to Sobolev spaces in several dimensions, since there are some differences with respect to Sobolev spaces in one spatial dimensional domain, which are the spaces used in the previous chapters.

Definition 8.1. A bounded domain Ω in $\mathbb{R}^d$ with boundary Γ is said to be of class C^N if there exists a family of bounded open sets

$$\mathcal{O}_j, \quad (j = 0, 1, \dots, r)$$

covering $\overline{\Omega}$ such that $\cup_{j=1}^{r}\mathcal{O}_j \supset \Gamma$ and for each $j = 1, \dots, r$ there exists a diffeomorphism φ_j (φ_j and φ_j^{-1} of class C^N) which sends $\mathcal{O}_j$ into the open unit ball $\mathcal{Q}$ in $\mathbb{R}^d$. φ_j sends

$$\begin{aligned} \mathcal{O}_j \cap \Omega \ \text{ into } \ \mathcal{Q}_+ &= \{(y', y_d) \in \mathcal{Q} \ | \ y_d > 0\} \\ \mathcal{O}_j \cap \Gamma \ \text{ into } \ \mathcal{Q}_0 &= \{(y', y_d) \in \mathcal{Q} \ | \ y_d = 0\} \\ \mathcal{O}_j \cap \complement\Omega \ \text{ into } \ \mathcal{Q}_- &= \{(y', y_d) \in \mathcal{Q} \ | \ y_n < 0\}, \end{aligned}$$

where $\complement\Omega$ denotes the complement of Ω.

If $\mathcal{O}_i \cap \mathcal{O}_j \neq \varnothing$, then there exists a homeomorphism J_{ij} of class C^N, with strictly positive Jacobian, of $\varphi_i(\mathcal{O}_i \cap \mathcal{O}_j)$ onto $\varphi_j(\mathcal{O}_i \cap \mathcal{O}_j)$ such that

$$\varphi_j(x) = J_{ij}(\varphi_i(x)), \quad \forall\, x \in \mathcal{O}_i \cap \mathcal{O}_j.$$

$\{\mathcal{O}_i, \varphi_i\}$, $j = 1, \dots, r$, is then a system of local maps defining Γ. We call $\{\alpha_i\}$ a partition of unity subordinate to the covering $\{\mathcal{O}_i \cap \overline{\Omega}\}$ of $\overline{\Omega}$ if

$$\alpha_i \in \mathcal{D}(\mathbb{R}^d), \quad \operatorname{supp} \alpha_i \subset \mathcal{O}_i \cap \overline{\Omega}, \quad \sum_{i=0}^{r} \alpha_i(x) = 1, \, \forall\, x \in \Omega.$$

If it is said that the domain has *smooth boundary*, then it is meant that the domain is of class C^∞. ♣

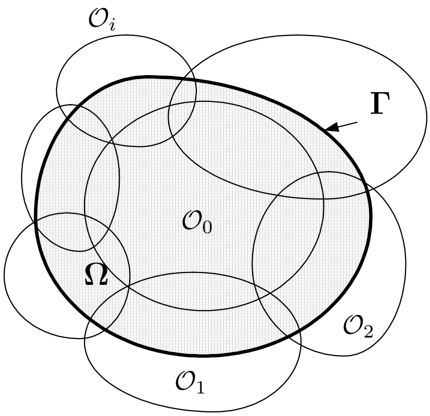

Figure 8.1.: Cover of Ω.

Let Ω be a bounded open set in $\mathbb{R}^d$ with *smooth* boundary Γ (sometimes denoted as $\partial\Omega$). Here $\mathcal{D}(\Omega)$ is the space of all indefinitely differentiable functions with a compact support in Ω. By $\mathcal{D}(\overline{\Omega})$ we denote the set

$$\mathcal{D}(\overline{\Omega}) = \{\phi \in \mathcal{D}(\mathbb{R}^d) \mid \phi_{|\Omega}\}, \tag{8.1}$$

or equivalently, if $\mathcal{O}$ denotes any open subset of $\mathbb{R}^d$ such that $\overline{\Omega} \subset \mathcal{O}$,

$$\mathcal{D}(\overline{\Omega}) = \{\phi \in \mathcal{D}(\mathcal{O}) \mid \phi_{|\Omega}\}.$$

Note that in this case we have $\mathcal{D}(\Omega) \subset \mathcal{D}(\overline{\Omega})$. Now let $\mathcal{D}'(\Omega)$ denote the dual space of the space of $\mathcal{D}(\Omega)$, often called the space of distributions on Ω, i.e., the set of continuous linear forms on $\mathcal{D}(\Omega)$: $\mathcal{D}'(\Omega) := \mathcal{L}(\mathcal{D}(\Omega))$. We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the duality between $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ and we remark that when f is a locally integrable function, then f can be identified with a distribution by

$$\tilde{f}(\phi) = \langle\langle f, \phi \rangle\rangle = \int_\Omega f(x)\phi(x)\, dx \quad \forall \phi \in \mathcal{D}(\Omega).$$

If we want to define a generalized derivative of a distribution so that for each $f \in \mathcal{D}(\Omega)$ we have

$$\partial \tilde{f}(\phi) = -\int_\Omega f \cdot \partial\phi\, dx = -\tilde{f}(\partial\phi), \quad \phi \in \mathcal{D}(\Omega),$$

then we must define ∂ as follows.

Definition 8.2 (page 4, [Sho97]). For each distribution u in $\mathcal{D}'(\Omega)$ the derivative $\partial u \in \mathcal{D}'(\Omega)$ is defined by

$$\partial \tilde{u}(\phi) = -\tilde{u}(\partial\phi), \quad \phi \in \mathcal{D}(\Omega).$$

♣

In general, we have for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^d$ and $|\alpha| = \sum_{i=1}^d \alpha_i$ that

Definition 8.3 (Section 1.1, [GR86]). For each distribution $u \in \mathcal{D}'(\Omega)$, we define $\partial^\alpha u$ in $\mathcal{D}'(\Omega)$ by

$$\partial^\alpha \tilde{u} = \langle\langle \partial^\alpha u, \phi \rangle\rangle = (-1)^{|\alpha|} \langle\langle u, \partial^\alpha \phi \rangle\rangle, \quad \forall \phi \in \mathcal{D}(\Omega),$$

i.e., if $u \in \mathcal{C}^{|\alpha|}(\overline{\Omega})$, then

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}.$$

♣

For $m \in \mathbb{N}$ we define the *Sobolev space*

$$H^m(\Omega) = \{v \in L_2(\Omega) \mid \partial^\alpha v \in L_2(\Omega),\ \forall |\alpha| \leq m\}, \tag{8.2}$$

which is a Hilbert space for the norm

$$\|u\|_{m,\Omega} = \left(\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha u(x)|^2 \, dx \right)^{1/2}. \tag{8.3}$$

This is a Hilbert space for the inner product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega \partial^\alpha u(x) \partial^\alpha v(x) \, dx. \tag{8.4}$$

We will also use the notation $\langle u, v \rangle_{m,\Omega} = \langle u, v \rangle_{H^m(\Omega)}$.

Since $\mathcal{D}(\Omega) \subset H^m(\Omega)$, we define the space $H_0^m(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ for the norm $\|\cdot\|_{H^m(\Omega)}$. When $m \geq 1$ and Ω is a proper subset of $\mathbb{R}^d$ then $H_0^m(\Omega)$ is generally a proper subspace of $H^m(\Omega)$. On the other hand, when $m = 0$ we have the following result.

Lemma 8.4 (Lemma 1.1 of [GR86]): *Let Ω be a bounded open set in $\mathbb{R}^d$ with a sufficiently regular boundary Γ. Then the space $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$.* ♡

We denote by $\|v\|$ the norm in $L_2(\Omega)$ ($L_2(\Omega)^n$) of an element $v \in L_2(\Omega)$ (respectively $v \in L_2(\Omega)^n$) and by $\langle v, u \rangle$ the inner product of two elements v, u in these spaces. Similarly, $\|v\|_\Gamma$ denotes the norm in $L_2(\Gamma)$ ($L_2(\Gamma)^n$) of an element $v \in L_2(\Gamma)$ (respectively $v \in L_2(\Gamma)^n$) and by $\langle v, u \rangle_\Gamma$ the inner product of two elements v, u in these spaces. Sometimes we will write $\langle \cdot, \cdot \rangle_{L_2(\Gamma)^m}$ to emphasize that it is the inner product on $L_2(\Gamma)^m$. Also, $\langle\langle \cdot, \cdot \rangle\rangle_H$ denotes the duality product of H and its dual.

The next theorem shows that smooth functions are dense in $H^m(\Omega)$.

Theorem 8.5 (Th. 1.2 of [GR86]): *Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary.*

1) *The space $\mathcal{D}(\overline{\Omega})$ is dense in $H^m(\Omega)$ for all integers $m \geq 0$.*
2) *Let $u \in H^m(\Omega)$ and let $\tilde{u}$ denote its extension by zero outside Ω. If $\tilde{u} \in H^m(\mathbb{R}^d)$ then $u \in H_0^m(\Omega)$.*
3) *If in addition Γ is bounded and $m \geq 1$, there exists a continuous linear extension operator P from $H^m(\Omega)$ into $H^m(\mathbb{R}^d)$:*

$$Pu|_\Omega = u \quad \forall\, u \in H^m(\Omega).$$

Let $d\sigma$ denote the *surface measure* on Γ and let $L_2(\Gamma)$ be the space of square integrable functions on Γ with respect to $d\sigma$, equipped with the norm

$$\|v\|_{L_2(\Gamma)} = \left(\int_\Gamma (v(\sigma))^2 \, d\sigma \right)^{1/2}.$$

Theorem 8.6 (Th. 1.2 of [GR86]): *Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary Γ.*

- *$\mathcal{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$.*
- *The space $\mathcal{D}(\Gamma)$ is dense in $H^s(\Gamma)$, $s \geq 0$.*
- *There exists a constant C such that*

$$\|\gamma_0 \, \phi\|_{L_2(\Gamma)} \leq C \, \|\phi\|_{H^1(\Omega)}, \quad \forall \phi \in \mathcal{D}(\overline{\Omega}), \tag{8.5}$$

where $\gamma_0 \, \phi$ denotes the value of ϕ on Γ.

It thus follows that the mapping γ_0 defined on $\mathcal{D}(\overline{\Omega})$ can be extended by continuity to a mapping, still called γ_0 (*trace of order zero*), from $H^1(\Omega)$ into $L_2(\Gamma)$, i.e., $\gamma_0 \in \mathcal{L}(H^1(\Omega); \, L_2(\Gamma))$.

Theorem 8.7 (Th. 1.3 of [GR86]): *Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary Γ. Then*

- $\ker(\gamma_0) = H_0^1(\Omega)$.
- *The range space of $\gamma_0 : H^1(\Omega) \to L_2(\Gamma)$ is a proper and dense subspace of $L_2(\Gamma)$, called $H^{1/2}(\Gamma)$.*

In the particular case $\Omega = \mathbb{R}^d$, it is possible to give an equivalent definition of $H^m(\Omega)$, by making use of the Fourier transform. If $u \in L_2(\mathbb{R}^d)$, the *Fourier transform* $\hat{u}$ in $L_2(\mathbb{R}^d)$ is defined by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-i \, x \cdot \xi) u(x) \, dx, \quad x \cdot \xi = x_1 \xi_1 + \cdots + x_d \xi_d, \tag{8.6}$$

the integral converging in the sense of L_2 and $u \to \hat{u}$ is an isomorphism of $L_2(\mathbb{R}^d)$ onto $L_2(\mathbb{R}^d)$. We set

$$\hat{u} = \mathcal{F} u, \quad \text{and} \quad u = \overline{\mathcal{F}} \hat{u} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(i \, x \cdot \xi) \hat{u}(\xi) \, d\xi.$$

It is well-known that

$$\begin{aligned} \mathcal{F}(\partial^\alpha u) &= (i\,\xi)^\alpha \mathcal{F}u, \quad \forall\,\alpha \\ \partial^\beta \mathcal{F}(u) &= \mathcal{F}\big((i\,x)^\beta u\big), \quad \forall\,\beta. \end{aligned} \tag{8.7}$$

Thus, if $\Omega = \mathbb{R}^d$, $H^m(\Omega)$ can be described by (8.2) or by

$$H^m(\mathbb{R}^d) = \{v \in L_2(\mathbb{R}^d) \mid (1+|\xi|^2)^{m/2}\hat{v} \in L_2(\mathbb{R}^d)\}, \tag{8.8}$$

with the norm

$$\|v\|_{H^m(\mathbb{R}^d)} = \left\| (1+|\xi|^2)^{m/2}\hat{v} \right\|_{L_2(\mathbb{R}^d)} \tag{8.9}$$

being equivalent to the norm (8.3). It is worth to mention that the representation (8.8) is also valid for $m \in \mathbb{R}$, see [LM72].

8.2. Some auxiliary spaces

In this chapter we consider the operator $\mathcal{J}$ defined by

$$\mathcal{J}e = P_0 e + \sum_{j=1}^{d} P_j \frac{\partial e}{\partial x_j}. \tag{8.10}$$

where

$$P_0^T = -P_0 \quad \text{and} \quad P_j^T = P_j, \quad j = 1, \dots, d. \tag{8.11}$$

Following the notation of previous chapters we should write $P_{1,j}$ since this is a first-order differential operator. However, for simplicity we denote them by P_j. As we did in the previous chapters, we start by studying Stokes theorem applied to this operator $\mathcal{J}$. For the moment let $e \in H^1(\Omega, \mathbb{R}^n)$ and Ω be a bounded open subset of $\mathbb{R}^d$ with a smooth boundary Γ. Next we study the expression $\mathcal{J}e_1 \cdot e_2 + e_1 \cdot \mathcal{J}e_2$ (the dot denotes the dot product on $\mathbb{R}^n$) and we show that it can be written in divergence form. Observe that we have for $e_1,\, e_2 \in H^1(\Omega, \mathbb{R}^n)$

$$\begin{aligned} \mathcal{J}e_1 \cdot e_2 + e_1 \cdot \mathcal{J}e_2 &= \left(P_0 e_1 + \sum_{j=1}^{d} P_j \frac{\partial e_1}{\partial x_j} \right) \cdot e_2 + e_1 \cdot \left(P_0 e_2 + \sum_{j=1}^{d} P_j \frac{\partial e_2}{\partial x_j} \right) \\ &= P_0 e_1 \cdot e_2 + e_1 \cdot P_0 e_2 + \sum_{j=1}^{d} P_j \frac{\partial e_1}{\partial x_j} \cdot e_2 + \sum_{j=1}^{d} e_1 \cdot P_j \frac{\partial e_2}{\partial x_j} \end{aligned}$$

Using equation (8.11) we obtain

$$\mathcal{J}e_1 \cdot e_2 + e_1 \cdot \mathcal{J}e_2 = \sum_{j=1}^{d} \left(P_j \frac{\partial e_1}{\partial x_j} \cdot e_2 + e_1 \cdot P_j \frac{\partial e_2}{\partial x_j} \right) = \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (e_1 \cdot P_j e_2)$$

$$= \operatorname{div} \begin{bmatrix} e_1 \cdot P_1 e_2 \\ \vdots \\ e_1 \cdot P_d e_2 \end{bmatrix}. \tag{8.12}$$

Theorem 8.8: *Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary Γ and $\mathcal{J}$ be a skew symmetric operator defined by* (8.10)–(8.11) *with $D(\mathcal{J}) = H^1(\Omega, \mathbb{R}^n)$. Then for any two functions $e_i \in D(\mathcal{J})$, $i \in \{1, 2\}$, we have that*

$$\langle \mathcal{J}e_1, e_2 \rangle_{L_2(\Omega)^n} + \langle e_1, \mathcal{J}e_2 \rangle_{L_2(\Omega)^n} = \left\langle e_{1_{|\Gamma}}, Q_\eta e_{2_{|\Gamma}} \right\rangle_{L_2(\Gamma)^n}, \quad \forall\, e_1,\, e_2 \in H^1(\Omega)^n, \tag{8.13}$$

where $e_{i_{|\Gamma}} = \gamma_0(e_i)$ is the restriction of e_i to Γ, the symmetric operator Q_η is defined by

$$Q_\eta = \eta_1 P_1 + \cdots + \eta_d P_d \tag{8.14}$$

and $\eta = [\eta_1,\, \ldots,\, \eta_d]^T$ is the outward unit normal vector field on Γ.

PROOF: Following (8.12) we write

$$\langle \mathcal{J}e_1, e_2 \rangle_{L_2(\Omega)^n} + \langle e_1, \mathcal{J}e_2 \rangle_{L_2(\Omega)^n} = \int_\Omega (\mathcal{J}e_1 \cdot e_2 + e_1 \cdot \mathcal{J}e_2)\, d\omega$$

$$= \int_\Omega \operatorname{div} \begin{bmatrix} e_1 \cdot P_1 e_2 \\ \vdots \\ e_1 \cdot P_d e_2 \end{bmatrix} d\omega.$$

Since $e_i \in H^1(\Omega)^n$, $i \in \{1, 2\}$, Gauss's divergence theorem applies and we thus obtain

$$\langle \mathcal{J}e_1, e_2 \rangle_{L_2(\Omega)^n} + \langle e_1, \mathcal{J}e_2 \rangle_{L_2(\Omega)^n} = \int_\Gamma \begin{bmatrix} e_1 \cdot P_1 e_2 \\ \vdots \\ e_1 \cdot P_d e_2 \end{bmatrix}_{|\Gamma} \cdot \eta\, d\omega,$$

where η is the outward unit normal vector field on Γ. Letting η_i be the i-th component of η we can rewrite the equation above as follows

$$\langle \mathcal{J}e_1, e_2 \rangle_{L_2(\Omega)^n} + \langle e_1, \mathcal{J}e_2 \rangle_{L_2(\Omega)^n} = \int_\Gamma e_1 \cdot (\eta_1 P_1 + \cdots + \eta_d P_d)\, e_2\, d\omega$$

From this the result follows. Since the matrices P_j, $j = 1, \ldots, d$, are symmetric it is easy to see that Q_η is a symmetric operator. ∎

Observe that Q_η, see (8.14), considered as an operator on $L_2(\Gamma)^n$ is a continuous operator since the boundary Γ is assumed to be smooth. Note also that the theorem above can be seen as Stokes theorem applied to the formally skew-symmetric operator $\mathcal{J}$, and equation (8.13) can be seen as a Green's identity. As it was done in previous chapters we could use this as a starting point to define the boundary port-variables. However, as we did in Section 6.5, we have to define the assumptions we make on the matrices P_j. In this chapter we make the following assumptions.

ASSUMPTION 8.9: a) The boundary of the bounded open set $\Omega \subset \mathbb{R}^d$ is characteristic with constant multiplicity. This means that $\dim\ker Q_\eta(x)$ is constant on each component of the boundary of Ω, i.e.,

$$\begin{aligned} &\dim\ker Q_\eta(x) = d - 2r, \qquad \forall\, x \in \Gamma, \\ \iff\ &\text{rank } Q_\eta(x) = 2r, \end{aligned}$$

where $d > 2r \in \mathbb{N}$ is constant.

b) The spectrum of $Q_\eta(x)$, $x \in \Gamma$, is symmetric with respect to the imaginary axis, and the sign of its eigenvalues does not change for all $x \in \Gamma$. That is, there are r eigenvalues with positive eigenvalues. ♡

Now we can try to follow the ideas presented in Section 6.5. Next we proceed to define the port variables. Following Assumption 8.9 and the smoothness on the boundary of Γ, we have that the eigenvalues of Q_η are continuous real-valued functions and Q_η can also be diagonalized. That is, there exist a unitary operator $\widetilde{R}(x)$ and a diagonal matrix containing the eigenvalues of Q_η, say Λ, such that

$$Q_\eta = \widetilde{R}\,\Lambda\,\widetilde{R}^*, \quad \text{with} \quad \widetilde{R}^*\,\widetilde{R} = I_{L_2(\Gamma)^n}, \quad \Lambda = \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & -\Lambda_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8.15}$$

Here, $I_{L_2(\Gamma)^n}$ is the identity operator on $L_2(\Gamma)^n$ (also denoted by I) and $\Lambda_1 \in \mathcal{L}(L_2(\Gamma)^r)$ contains the positive eigenvalues of Q_η, which is possible by Assumption 8.9. Now observe that Λ satisfies

$$\begin{aligned} \left[\begin{array}{cc|c} \Lambda_1 & 0 & 0 \\ 0 & -\Lambda_1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] &= \frac{1}{2}\begin{bmatrix} \Lambda_1 & -\Lambda_1 & 0 \\ I & I & 0 \\ 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \Lambda_1 & -\Lambda_1 & 0 \\ I & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R_{\text{ext}} & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} R_{\text{ext}} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \left[\begin{array}{c|c} R_{\text{ext}}^*\Sigma\, R_{\text{ext}} & 0 \\ \hline 0 & 0 \end{array}\right], \end{aligned} \tag{8.16}$$

where I is the identity operator in the respective space, and

$$R_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \Lambda_1 & -\Lambda_1 \\ I & I \end{bmatrix} \in \mathcal{L}(L_2(\Gamma)^{2r}), \quad \Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathcal{L}(L_2(\Gamma)^{2r}). \tag{8.17}$$

Accordingly to (8.16) we partition the unitary operator $\widetilde{R} \in \mathcal{L}(L_2(\Gamma)^n)$ as follows

$$\widetilde{R}^* = \left[\frac{R^*}{R_0^*}\right], \quad R^* \in \mathcal{L}(L_2(\Gamma)^n,\, L_2(\Gamma)^{2r}). \tag{8.18}$$

Note that R^* could be considered as a projection onto the range of Q_η and R_0^* a projection onto the kernel of Q_η.

Definition 8.10. Let $r \in \mathbb{N}$ be as described in Assumption 8.9 and $\gamma_0 : H^1(\Omega)^n \to L_2(\Gamma)^n$ be the trace operator of order zero, i.e., $\gamma_0(u) = u_{|\Gamma}$ for $u \in H^1(\Omega)^n$. Then, the *boundary port-variables* associated with the differential operator $\mathcal{J}$ are the operators e_∂, $f_\partial \in \mathcal{L}(H^1(\Omega)^n,\, L_2(\Gamma)^r)$, defined by

$$\begin{bmatrix} f_{\partial,u} \\ e_{\partial,u} \end{bmatrix} = R_{\text{ext}}\, R^*\, \gamma_0(u), \qquad u \in H^1(\Omega)^n, \quad R^*\gamma_0(u) \in L_2(\Gamma)^{2r} \tag{8.19}$$

where R_{ext} and R^* are defined by (8.17) and (8.18), respectively. ♣

Now we can reformulate Theorem 8.8 to include the port variables. This follows from the following theorem.

Theorem 8.11: *Let Ω be a bounded open set in $\mathbb{R}^d$ with a smooth boundary Γ. Then Green's identity* (8.13) *can be rewritten as*

$$\langle \mathcal{J}u, v\rangle_{L_2(\Omega)^n} + \langle u, \mathcal{J}v\rangle_{L_2(\Omega)^n} = \langle f_{\partial,v}, e_{\partial,u}\rangle_{L_2(\Gamma)^r} + \langle e_{\partial,v}, f_{\partial,u}\rangle_{L_2(\Gamma)^r} \tag{8.20}$$

for all $v,\, u \in H^1(\Omega)^n$. Furthermore, the range of the operator $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] : H^1(\Omega)^n \to L_2(\Gamma)^{2r}$ is dense in $L_2(\Gamma)^{2r}$.

PROOF: The first part follows easily from (8.15)–(8.19). Indeed,

$$\begin{aligned} \langle Q_\eta v, u\rangle_{L_2(\Gamma)^n} &= \left\langle \Lambda\, \widetilde{R}^* v, \widetilde{R}^*\, u \right\rangle_{L_2(\Gamma)^n} = \langle \Sigma\, R_{\text{ext}}\, R^* v, R_{\text{ext}}\, R^*\, u\rangle_{L_2(\Gamma)^{2r}} \\ &= \left\langle \Sigma \begin{bmatrix} f_{\partial,v} \\ e_{\partial,v} \end{bmatrix}, \begin{bmatrix} f_{\partial,u} \\ e_{\partial,u} \end{bmatrix} \right\rangle_{L_2(\Gamma)^{2r}}. \end{aligned}$$

The denseness of ran $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ follows from Theorem 8.7 and the surjectivity of the operator $R_{\text{ext}}\, R^*$. ■

Since the range of the boundary port operator $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ is a proper subset of $L_2(\Gamma)^n$, called $H^{1/2}(\Gamma)$, we would like to consider the *maximal domain* of the operator $\mathcal{J}$, see (6.56). So we consider the space $H(J,\Omega)$ defined as

$$H(J,\Omega) = \{v \in L_2(\Omega)^n \mid \mathcal{J}v \in L_2(\Omega)^n\}. \tag{8.21}$$

This is a Hilbert space, see [Rau85], when endowed with the norm

$$\|v\|_{\mathcal{J}}^2 = \|v\|^2 + \|\mathcal{J}v\|^2. \tag{8.22}$$

Recall that in Section 6.5 we had that the boundary operator was surjective for all functions on $H^N(a,b)$, see Theorem 2.6, and hence for all functions on $H(J,(a,b))$. However, when the dimension of the spatial domain is larger than one, this is not always true, see Theorem 8.11. Thus, one may hope that by extending the boundary operator to the larger domain $H(J,\Omega)$ one may obtain surjectivity, which is needed in order to define boundary control systems, see Definition 1.10.b. But, as the following example shows, this is still not true.

Example 8.12 Consider a plane membrane with a smooth shape, homogeneously stretched by a tension T and with a mass density ρ. Small vibrations on the membrane are described by the $2D$ wave equation

$$\frac{\partial^2\nu}{\partial t^2} = \frac{T}{\rho}\left(\frac{\partial^2\nu}{\partial x_1^2} + \frac{\partial^2\nu}{\partial x_2^2}\right), \tag{8.23}$$

where $\nu(x_1,x_2,t)$ is the amplitude at position (x_1,x_2) and time t. This can be seen as a model for the vibration of the head of a drum. For simplicity we assume $T=1$, $\rho=1$, and Ω is bounded (e.g. the unit circle) with smooth boundary Γ. Equation (8.23) can be written as a port-Hamiltonian system by introducing the (energy) variables

$$s = \frac{\partial\nu}{\partial t}, \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial\nu}{\partial x} \\ \frac{\partial\nu}{\partial y} \end{bmatrix} = \nabla\,\nu. \tag{8.24}$$

This gives that equation (8.23) becomes

$$\begin{aligned} \frac{\partial s}{\partial t} &= \nabla\cdot v = \operatorname{div}(v) \\ \frac{\partial v}{\partial t} &= \nabla s, \end{aligned} \tag{8.25}$$

where ∇ is the gradient and div is the divergence operator, see [DL85b, Chapter IX]. Observe that $\mathcal{J}$ is (with an abuse of notation) $\left[\begin{smallmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{smallmatrix}\right]$, which can also be written as

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{bmatrix} s \\ v_1 \\ v_2 \end{bmatrix} =& \frac{\partial}{\partial x}\begin{bmatrix} v_1 \\ s \\ 0 \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} v_2 \\ 0 \\ s \end{bmatrix} \\
=& \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x_1}\begin{bmatrix} s \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x_2}\begin{bmatrix} s \\ v_1 \\ v_2 \end{bmatrix} \\
=& P_1 \frac{\partial}{\partial x_1}\begin{bmatrix} s \\ v_1 \\ v_2 \end{bmatrix} + P_2 \frac{\partial}{\partial x_2}\begin{bmatrix} s \\ v_1 \\ v_2 \end{bmatrix}.
\end{aligned} \tag{8.26}
$$

For this system we have, see (8.14),

$$
Q_\eta = \begin{pmatrix} 0 & \eta_1 & \eta_2 \\ \eta_1 & 0 & 0 \\ \eta_2 & 0 & 0 \end{pmatrix}. \tag{8.27}
$$

Next we want to find the port-variables. For this we choose R, which satisfies $R^* R = I$, as follows (recall that η is the unit normal)

$$
R = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ \eta_1 & \eta_1 \\ \eta_2 & \eta_2 \end{pmatrix}, \quad \text{and thus} \quad \begin{bmatrix} \Lambda_1 & 0 \\ 0 & -\Lambda_1 \end{bmatrix} = R^* Q_\eta R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

From (8.17) we get that

$$
R_{\text{ext}} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},
$$

and hence the ports become

$$
\begin{bmatrix} f_{\partial,w} \\ e_{\partial,w} \end{bmatrix} = R_{\text{ext}} R^* \begin{bmatrix} s \\ v \end{bmatrix}_{|\Gamma} = \begin{bmatrix} s \\ \eta \cdot v \end{bmatrix}_{|\Gamma}, \qquad w = \begin{bmatrix} s \\ v \end{bmatrix}. \tag{8.28}
$$

Note, from (8.24), that the term $\eta \cdot v$ corresponds to $\frac{\partial \nu}{\partial \eta}$, where $\frac{\partial}{\partial \eta} = \eta \cdot \nabla$ is the outward normal derivative. It is worth to mention that these are the typical boundary variables for the $2D$ wave equation. Next we proceed to find the range of the boundary port operator. To do this, first note that the maximal domain of $\mathcal{J}$, see (8.21), is given by

$$
\begin{aligned}
H(J,\Omega) =& \left\{ \begin{bmatrix} s \\ v_1 \\ v_2 \end{bmatrix} \in L_2(\Omega)^3 \;\middle|\; \frac{\partial s}{\partial x_1}, \frac{\partial s}{\partial x_2}, \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) \in L_2(\Omega) \right\} \\
=& \left\{ \begin{bmatrix} s \\ v_1 \\ v_2 \end{bmatrix} \in L_2(\Omega)^3 \;\middle|\; s \in H^1(\Omega), \text{ and } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in H(\text{div}\,,\Omega) \right\},
\end{aligned}
$$

where $H(\text{div}\,,\Omega) = \{v \in L_2(\Omega)^2 \mid \text{div}\,(v) \in L_2(\Omega)\}$. Since $s \in H^1(\Omega)$ we have, from Theorem 8.7, that the range of f_∂, see (8.28), is equal to $H^{1/2}(\Gamma) \subset L_2(\Gamma)$. On the other hand, it is know, see Theorem 2.5 and Corollary 2.8 of [GR86], that the boundary mapping $\gamma_n : v \rightarrow \eta \cdot v_{|\Gamma}$ defined on $H(\text{div}\,,\Omega)$ is surjective onto $H^{-1/2}(\Gamma)$, which is the dual space of $H^{1/2}(\Gamma)$. Summarizing we have for the $2D$ wave equation that

$$\text{ran} \begin{bmatrix} f_{\partial,w} \\ e_{\partial,w} \end{bmatrix} = \begin{bmatrix} H^{1/2}(\Gamma) \\ H^{-1/2}(\Gamma) \end{bmatrix}, \quad \forall\, w \in H(J,\Omega). \tag{8.29}$$

*

The example above shows that the selection of the input and output spaces for $2D$ and $3D$ systems is not as trivial as it is for the $1D$ case. To make things worse, the operators f_∂ and e_∂ do not even map the same space. One may also wonder whether, in general, the boundary port operator is surjective as in (8.29). But this is not the case. In fact, for Maxwell equations one can show that the boundary port operator is not surjective in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, see [Tar97] or [GR86].

In the example above we can see that the boundary port operator can also be defined on $H(J,\Omega)$. Below we describe some difficulties that arise when one tries to extend this operator in general.

Following the definition of the space $H(J,\Omega)$ given in (8.21) we also consider the space $H_0(J,\Omega)$ defined as the closure of $\mathcal{D}(\Omega)^n$ in the space $H(J,\Omega)$, i.e.,

$$H_0(J,\Omega) = \overline{\mathcal{D}(\Omega)^n}^{H(J,\Omega)}. \tag{8.30}$$

The following results are useful for proving results for the operator $\mathcal{J}$.

Lemma 8.13: *Let Ω be a bounded open set in $\mathbb{R}^d$ with a Lipschitz boundary Γ. Let $v \in H(J,\Omega)$ be such that*

$$\langle \mathcal{J}v, \phi \rangle + \langle v, \mathcal{J}\phi \rangle = 0, \qquad \forall\, \phi \in \mathcal{D}(\overline{\Omega})^n. \tag{8.31}$$

Then, $v \in H_0(J,\Omega)$. ♡

PROOF: We use the same idea of the proof of Lemma 1 of [DL85b, p.206]. Basically, we want to prove something similar to point 2 in Theorem 8.5. Let $v_0 := \mathcal{J}v$, and denote by $\tilde{v}$ and $\tilde{v}_0$ the extension of v and v_0 to $\mathbb{R}^d$, which are zero outside Ω. Equation (8.31) implies that

$$\langle \tilde{v}_0, \theta \rangle_{L_2(\mathbb{R}^d)} + \langle \tilde{v}, \mathcal{J}\theta \rangle_{L_2(\mathbb{R}^d)} = 0, \qquad \forall\, \theta \in \mathcal{D}(\mathbb{R}^d)^n.$$

However, since this holds for all θ with compact support, we can see by, using the derivative in the distributional sense (see Definition 8.3) in $\langle \tilde{v}, \mathcal{J}\theta \rangle$, that $\tilde{v}$ satisfies $\langle\langle \tilde{v}, \mathcal{J}\theta \rangle\rangle_{\mathcal{D}(\mathbb{R}^d)^n} = -\langle\langle \mathcal{J}\tilde{v}, \theta \rangle\rangle_{\mathcal{D}(\mathbb{R}^d)^n}$. Hence, the equation above becomes

$$\langle\langle \tilde{v}_0, \theta \rangle\rangle_{\mathcal{D}(\mathbb{R}^d)^n} + \langle\langle -\mathcal{J}\tilde{v}, \theta \rangle\rangle_{\mathcal{D}(\mathbb{R}^d)^n} = 0, \qquad \forall\, \theta \in \mathcal{D}(\mathbb{R}^d)^n.$$

This in turn implies that $\tilde{v}_0 = \mathcal{J}\tilde{v}$ in $\mathcal{D}'(\mathbb{R}^d)^n$. Since $\tilde{v}_0 \in L_2(\mathbb{R}^d)^n$, we can see that $\tilde{v} \in H(J, \mathbb{R}^d)$ with $\operatorname{supp} \tilde{v} \subset \overline{\Omega}$.

It then suffices to construct a sequence in $\mathcal{D}(\mathbb{R}^d)^n$ that converges to $\tilde{v}$ in $H(J, \mathbb{R}^d)$ with supports in Ω. In other words, to find a sequence in $\mathcal{D}(\Omega)^n$ converging to $v \in H(J, \Omega)$. To do this, we will use a truncation and regularization argument, see [DL85a, pp. 457–463].

- If Ω is a bounded open set which is strictly star-like[1], then the family of functions $\tilde{v}_\sigma$ defined for all $\sigma \in [0, 1)$ by

  $$\tilde{v}_\sigma(x) = \tilde{v}\left(\frac{x}{\sigma}\right) \qquad \text{(hence with } \operatorname{supp} \tilde{v}_\sigma \subset \Omega\text{)}$$

  converges as $\sigma \uparrow 1$ to $\tilde{v} \in H(J, \mathbb{R}^d)$. Next, we regularize the functions $\tilde{v}_\sigma$. Let $\rho \in \mathcal{D}(\mathbb{R}^d)$ be such that $\rho \geq 0$, $\rho(x) = 0$ for $|x| \geq 1$ (i.e., $\operatorname{supp} \rho \subset B(0, 1)$), and $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$ (see [DL85a, Lemma 1,p.458]). Then the family of functions ρ_ϵ, $\epsilon > 0$, defined by (see [DL85a, pp.459–460]) $\rho_\epsilon(x) = \frac{1}{\epsilon^d}\rho\left(\frac{x}{\epsilon}\right)$ is such that

  $$\rho_\epsilon \in \mathcal{D}(\mathbb{R}^d), \quad \rho_\epsilon \geq 0, \quad \rho_\epsilon(x) = 0 \text{ for } |x| \geq \epsilon, \quad \int_{\mathbb{R}^d} \rho_\epsilon(x)\, dx = 1,$$

  which implies that for all $u \in L_2(\mathbb{R}^d)^n$, $\rho_\epsilon * u \to u$ in $L_2(\mathbb{R}^d)^n$, where the $*$ denotes the convolution product.

  Since $\tilde{v} \in H(J, \mathbb{R}^d)$ we have $\rho_\epsilon * \tilde{v} \to \tilde{v}$ in $H(J, \mathbb{R}^d)$ as $\epsilon \to 0$, and for ϵ sufficiently small $\operatorname{supp}(\rho_\epsilon * \tilde{v}_\sigma) \subset \Omega$, hence $\rho_\epsilon * \tilde{v}_\sigma \in \mathcal{D}(\Omega)^n$, see proof of Lemma 2 of [DL85a, p.460]. This establish the existence of a sequence $\rho_\epsilon * \tilde{v}_{\sigma_i}$ in $\mathcal{D}(\Omega)^n$ convergent to $\tilde{v}$ in $H(J, \Omega)$.

- The general case follows the same idea of point (iv) in the proof of Lemma 1 of [DL85b, p.206]. ∎

Theorem 8.14: *Let Ω be a bounded open set in $\mathbb{R}^d$ with a bounded, Lipschitz boundary Γ. Then the space $\mathcal{D}(\overline{\Omega})^n$ is dense in $H(J, \Omega)$.*

PROOF: Let $w \in H(J, \Omega)$ be orthogonal to $\mathcal{D}(\overline{\Omega})^n$; thus

$$\langle w, v \rangle_{\mathcal{J}} := \langle w, v \rangle + \langle \mathcal{J}w, \mathcal{J}v \rangle = 0, \quad \forall\, v \in \mathcal{D}(\overline{\Omega})^n. \tag{8.32}$$

[1] A bounded open set is star-like if there exists $y \in \Omega$ such that, $\sigma\overline{\Omega} \subset \Omega, \forall\, \sigma \in [0, 1)$ with respect to y (which we here take to be the origin)

Let $w_0 := \mathcal{J}w$. It is clear that $w_0 \in L_2(\Omega)^n$. Let $v \in \mathcal{D}(\Omega)^n$. Then we have that $\langle w_0, \mathcal{J}v\rangle = -\langle\langle \mathcal{J}w_0, v\rangle\rangle_{\mathcal{D}(\Omega)^n}$, which in the equation above gives

$$\langle\langle w, v\rangle\rangle_{\mathcal{D}(\Omega)^n} = \langle\langle \mathcal{J}w_0, v\rangle\rangle_{\mathcal{D}(\Omega)^n}\,, \quad \forall\, v \in \mathcal{D}(\Omega)^n.$$

This implies that $w = \mathcal{J}w_0$ in $\mathcal{D}'(\Omega)^n$. Since $w \in H(J,\Omega)$, we deduce that $w = \mathcal{J}w_0 \in L_2(\Omega)^n$ and thus $w_0 \in H(J,\Omega)$. Furthermore, it satisfies (from (8.32))

$$\langle \mathcal{J}w_0, v\rangle + \langle w_0, \mathcal{J}v\rangle = 0, \quad \forall\, v \in \mathcal{D}(\overline{\Omega})^n. \tag{8.33}$$

From Lemma 8.13 we conclude that $w_0 \in H_0(J,\Omega)$. Since $\mathcal{D}(\Omega)^n$ is dense in $H_0(J,\Omega)$ (by definition), there exits a sequence $\{\psi_k\}_{k\in\mathbb{N}}$, with $\psi_k \in \mathcal{D}(\Omega)^n$, which converges to w_0 in $H(J,\Omega)$. Thus, since $w_0 = \mathcal{J}w$ and $w = \mathcal{J}w_0$ we obtain

$$\begin{aligned}\langle w_0, w_0\rangle_{\mathcal{J}} &= \lim_{k\to\infty} \langle w_0, \psi_k\rangle_{\mathcal{J}} = \lim_{k\to\infty} \langle w_0, \psi_k\rangle + \langle \mathcal{J}w_0, \mathcal{J}\psi_k\rangle \\ &= \lim_{k\to\infty} \langle \mathcal{J}w, \psi_k\rangle + \langle w, \mathcal{J}\psi_k\rangle \\ &= \lim_{k\to\infty} \langle\langle \mathcal{J}w, \psi_k\rangle\rangle - \langle\langle \mathcal{J}w, \psi_k\rangle\rangle = 0.\end{aligned}$$

Hence w_0 equals zero and so does w. Therefore, $\mathcal{D}(\overline{\Omega})^n$ is dense in $H(J,\Omega)$. ■

Theorem 8.15: *Let Ω be a bounded open set in $\mathbb{R}^d$ with a smooth boundary Γ. The trace map $\gamma_Q : v \rightarrow Q_\eta v|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})^n$ extends by continuity to a continuous linear map – still denoted by γ_Q – from $H(J,\Omega)$ into $H^{-1/2}(\Gamma)^n$. Furthermore, Green's identity* (8.13) *holds for $v \in H(J,\Omega)$ and $u \in H^1(\Omega)^n$, that is*

$$\langle \mathcal{J}u, v\rangle + \langle u, \mathcal{J}v\rangle = \langle\langle Q_\eta v, u\rangle\rangle_{H^{1/2}(\Gamma)^n}\,, \quad \forall\, v \in H(J,\Omega),\; u \in H^1(\Omega)^n. \tag{8.34}$$

PROOF: We know that Green's formula (8.13) holds for all v and $u \in \mathcal{D}(\overline{\Omega})^n$, and also for all $u \in H^1(\Omega)$, that is

$$\langle \mathcal{J}u, v\rangle + \langle u, \mathcal{J}v\rangle = \int_\Gamma \langle Q_\eta v, u\rangle_{\mathbb{R}^n}\, d\gamma, \quad \forall\, v \in \mathcal{D}(\overline{\Omega})^n,\; u \in H^1(\Omega)^n. \tag{8.35}$$

Note that $\mathcal{J}$ satisfies $\|\mathcal{J}u\| \leq \tilde{c}\,\|u\|_{H^1(\Omega)^n}$ for any $u \in H^1(\Omega)$, and recall that $\|u\| \leq \|u\|_{H^1(\Omega)}$. We deduce from the equation above

$$\begin{aligned}\left|\int_\Gamma \langle Q_\eta v, u\rangle_{\mathbb{R}^n}\, d\gamma\right| &\leq \|\mathcal{J}u\|\,\|v\| + \|u\|\,\|\mathcal{J}v\| \\ &\leq \tilde{c}\,\|u\|_{H^1(\Omega)^n}\,\|v\| + \|u\|_{H^1(\Omega)^n}\,\|\mathcal{J}v\| \\ &\leq c\,\|u\|_{H^1(\Omega)^n}\,\|v\|_{\mathcal{J}}\,, \quad \forall\, v \in \mathcal{D}(\overline{\Omega})^n,\; u \in H^1(\Omega)^n\end{aligned}$$

where c and $\tilde{c}$ are real constants. Now, let μ be an element of $H^{1/2}(\Gamma)$. Then there exits an element $u \in H^1(\Omega)$ such that $u = \mu$ on Γ, see Theorem 5 of [DL85a, p.114]. Hence the above inequality implies that[2]

$$\left| \int_\Gamma \langle Q_\eta v, \mu \rangle_{\mathbb{R}^n} \, d\gamma \right| \leq c \, \|\mu\|_{H^{1/2}(\Gamma)^n} \, \|v\|_{\mathcal{J}} \,, \quad \forall \, v \in \mathcal{D}(\overline{\Omega})^n, \, \mu \in H^{1/2}(\Gamma)^n. \quad (8.36)$$

Thus

$$\|Q_\eta v\|_{H^{-1/2}(\Gamma)^n} \leq c \, \|v\|_{\mathcal{J}} \,.$$

It follows that the mapping $\gamma_Q \,:\, v \in \mathcal{D}(\overline{\Omega})^n \mapsto Q_\eta v|_\Gamma$ extends by continuity to a mapping from $H(J, \Omega)$ into $H^{-1/2}(\Gamma)^n$ (because of the denseness of $\mathcal{D}(\overline{\Omega})^n$ in $H(J, \Omega)$, from Theorem 8.14). Note that the inequality above holds for all $v \in H(J, \Omega)$ and that the Green's formula (8.35) is still true (by denseness) for all $v \in H(J, \Omega)$, $u \in H^1(\Omega)^n$, the integral on the right hand side of (8.35) being replaced by the duality product on $H^{1/2}(\Gamma)^n$, see (8.34). ■

We now look at the kernel of γ_Q in $H(J, \Omega)$.

Theorem 8.16: *The kernel* $\ker(\gamma_Q)$ *of* $\gamma_Q \,:\, H(J, \Omega) \;\to\; H^{-1/2}(\Gamma)$ *is the space* $H_0(J, \Omega)$*. That is*

$$H_0(J, \Omega) = \ker(\gamma_Q) = \{v \in H(J, \Omega) \mid Q_\eta v|_\Gamma = 0\}. \quad (8.37)$$

PROOF: By taking limits in (8.34), it is clear that $H_0(J, \Omega) \subset \ker(\gamma_Q)$ (see last part of the proof of Theorem 8.14).

Let $v \in \ker(\gamma_Q)$; then from (8.34), v is such that

$$\langle \mathcal{J} v, \phi \rangle + \langle v, \mathcal{J} \phi \rangle = 0, \qquad \forall \, \phi \in \mathcal{D}(\overline{\Omega})^n.$$

Then Lemma 8.13 implies that $v \in H_0(J, \Omega)$. Hence $\ker(\gamma_Q) \subset H_0(J, \Omega)$. ■

Theorem 8.15 confirms the difficulties that arise when dealing with systems in $2D$ or $3D$. In this case, instead of having an inner product on the boundary, we have a duality product. Although, if we recall from the general introduction of port-Hamiltonian systems in Section 1.7 the effort and flow spaces were selected to be dual to each other, see (1.32). So, from that point of view, the result of Theorem 8.15 is not that unexpected. Furthermore, Green's identity (8.34) does not hold for all elements in the same space. In fact, it is possible to extend (8.34) so that it is valid for all v, $u \in H(J, \Omega)$, but in that case the duality product

[2] Recall that $\|\mu\|_{H^{1/2}(\Gamma)} = \inf\limits_{\substack{u \in H^1(\Omega) \\ u|_\Gamma = \mu}} \|u\|_{H^1(\Omega)}$, see [GR86, p.8].

on the boundary extends to a space which does not have a useful elementary description, see [Rau85]. So, complications arise when one tries to extend the boundary operator to a larger space.

In order to continue with the discussion we make the following assumption based on Theorem 8.15 and Example 8.12.

ASSUMPTION 8.17: Let Ω be a bounded open set in $\mathbb{R}^d$ with a smooth boundary Γ. Assume that the boundary port operator $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ defined on $\mathcal{D}(\overline{\Omega})^n$ can be extended by continuity to a continuous linear map – still denoted by $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ – from $H(J,\Omega)$ into $H^{1/2}(\Gamma)^r \times H^{-1/2}(\Gamma)^r$. Furthermore, assume that Green's identity (8.20) holds for all $v, u \in H(J,\Omega)$, that is

$$\langle \mathcal{J}u, v\rangle + \langle u, \mathcal{J}v\rangle = \langle\langle e_{\partial,u}, f_{\partial,v}\rangle\rangle_{H^{1/2}(\Gamma)^r} + \langle\langle e_{\partial,v}, f_{\partial,u}\rangle\rangle_{H^{1/2}(\Gamma)^r}, \quad \forall\, v, u \in H(J,\Omega). \tag{8.38}$$

♡

Note that by restricting $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ to $H^1(\Omega)$ one obtains ran $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] = H^{1/2}(\Gamma)^{2r}$, see Theorem 8.7. Thus, under Assumption 8.17, one can see that ran $f_\partial = H^{1/2}(\Gamma)^r$ and ran $e_\partial \supset H^{1/2}(\Gamma)^r$. So, if Assumption 8.17 above is satisfied, then one could prove that by imposing a (boundary) condition on either f_∂ or e_∂ one obtains existence and uniqueness of solutions.

Theorem 8.18: *Let Assumption 8.17 be satisfied and consider the skew-symmetric operator $\mathcal{J}$ described in* (8.10)–(8.11). *Then the operator $\mathcal{A}$ defined by $\mathcal{A} = \mathcal{J}$ with domain*

$$D(\mathcal{A}) = \{x \in H(J,\Omega) \mid e_{\partial,x} = 0\} \tag{8.39}$$

is skew-adjoint, i.e., $\mathcal{A}^* = -\mathcal{A}$ and $D(\mathcal{A}^*) = D(\mathcal{A})$. As a consequence, $\mathcal{A}$ generates a unitary semigroup.

PROOF: First recall the definition of the adjoint operator. The adjoint operator is defined by

$$\mathcal{A}^* u = \{w \in L_2(\Omega)^n \mid \forall\, y \in D(\mathcal{A}) \text{ we have } \langle \mathcal{A}y, u\rangle = \langle y, w\rangle\}, \tag{8.40a}$$

with domain

$$u \in D(\mathcal{A}^*) \iff \exists\, w \in L_2(\Omega)^n \text{ s.t. } \forall\, y \in D(\mathcal{A}) \text{ we have } \langle \mathcal{A}y, u\rangle = \langle y, w\rangle. \tag{8.40b}$$

First we show that $D(\mathcal{A}^*) \subset H(J,\Omega)$. Let $u \in D(\mathcal{A}^*)$. This implies, from the definition of the adjoint operator, that there exist a $w \in L_2(\Omega)^n$ such that

$$\langle \mathcal{A}y, u\rangle = \langle y, w\rangle \qquad \forall\, y \in D(\mathcal{A}), \tag{8.41}$$

and $\mathcal{A}^*u = w$. Clearly, all $\mathcal{D}(\Omega)^n$ are in $D(\mathcal{A})$. Thus we can select any y with compact support in the equation above, which gives

$$\langle \mathcal{J}y, u\rangle = \langle y, w\rangle \qquad \forall\, y \in \mathcal{D}(\Omega)^n.$$

Since this holds for all $y \in \mathcal{D}(\Omega)^n$, we can see by using the derivative in the distributional sense, see Definition 8.3, that $\langle\langle \mathcal{J}\, y, u\rangle\rangle_{\mathcal{D}(a,b)} = -\,\langle\langle y, \mathcal{J}\, u\rangle\rangle_{\mathcal{D}(a,b)}$. Using this in the equation above yields that $w = -\mathcal{J}\, u$ in $\mathcal{D}'(\Omega)^n$. Since $w \in L_2(\Omega)^n$, we conclude that $w = -\mathcal{J}\, u \in L_2(\Omega)^n$, and hence $D(\mathcal{A}^*) \subset H(J, \Omega)$ (since $u \in D(\mathcal{A}^*)$ was arbitrary). So, we only need to prove the boundary condition. The argument above allows us to use (8.38). Thus, for all $y \in D(\mathcal{A})$ and $u \in D(\mathcal{A}^*)$ equation (8.38) becomes

$$\langle \mathcal{A}y, u\rangle = \langle y, -\mathcal{J}u\rangle + \langle\langle e_{\partial,u}, f_{\partial,y}\rangle\rangle_{H^{1/2}(\Gamma)^r}, \quad \forall\, y \in D(\mathcal{A}), \text{ and } \forall u \in D(\mathcal{A}^*).$$

By using the definition of the adjoint operator we conclude that $\mathcal{A}^* = -\mathcal{J}$ and $f_{\partial,y}$ must lie in the kernel of the functional $e_{\partial,u}$. Since $\operatorname{ran} f_\partial = H^{1/2}(\Gamma)^r$ (by assumption) we conclude that $e_{\partial,u} = 0$. This proves the result. ■

It is easy to see that in the case of the wave equation in Example 8.12, the theorem above applies. Also, in the case of the Mindlin plate, see [MMB05], one obtains the same result.

Summarizing, we can see that extending the ideas presented in the previous chapters to $2D$ and $3D$ systems is more difficult, mainly because the functions behave very differently on the boundary. However, we can see that there is a structure on the models that can help to extend the results presented in Chapter 2, and this structure is not too different to the $1D$ case. Thus, studying further the approach presented in this book to deal with $2D$ and $3D$ systems could lead to some interesting results and perhaps to a simplification of the theory.

Chapter 9

Conclusions and future work

In this thesis we have tried to provide a mathematical framework for the modeling and analysis of open distributed parameter systems. From a mathematical point of view this thesis merges the approach based on Hamiltonian modeling of open distributed-parameter systems, employing the notion of port-Hamiltonian systems, with the semigroup approach of infinite-dimensional systems theory. The Hamiltonian representation provides powerful analysis methods (e.g. for stability), and it enables the use of Lyapunov-stability theory and passivity-based control. The semigroup approach has been widely applied in the analysis of distributed parameter systems and it has facilitated the extension of some notions from finite-dimensional system theory to the infinite-dimensional case.

We have seen that the port-Hamiltonian approach could lead, in some cases, to a simplification of the theory and a better understanding of certain properties of distributed parameter systems by dealing with classes of systems based on the structure of the model provided. We have mainly used the port-Hamiltonian formulation for the analysis of 1D-boundary control systems. These are systems in which the input (or part of it) acts on the boundary of the spatial domain. In these cases it is possible to parameterize the selection of the inputs (boundary conditions) and outputs by the selection of two matrices in such a way that the resulting system is passive. In this case these matrices determine the supply rate of the passive system, making it easy, in particular, to obtain impedance passive and scattering passive systems. In fact, as it is shown, these matrices can be used to determine further properties of the system, such as stability, controllability, and observability. This could be a first step towards a "matrix theory" for linear distributed parameter systems on one-dimensional spatial domain.

As mentioned earlier, one of the key points of the port-Hamiltonian approach is that it allows to deal with classes of systems. These classes of systems depend on the structure of the model. This thesis has treated mainly two broad classes of systems. One corresponds to systems where the dissipation phenomena is not present and the other includes systems with some type of dissipation (e.g. heat or mass transfer, damping). It is shown that this already covers a very large class of 1D-systems. These classes can, in turn, be divided into subclasses according to the properties of the structure, to provide further tools for the analysis of such systems. From a modeling point of view, this structure of the model arises naturally, mainly because of the use of Hamiltonian modeling ideas.

This thesis has been mainly intended to show that the port-Hamiltonian formulation could be an interesting and useful approach in the analysis of distributed parameter systems. However, as a relative new approach for the analysis of DPS, it has its advantages and disadvantages when compared with other more commonly used approaches. The author hopes that at least the reader can see the port-Hamiltonian approach as another useful tool in the analysis of DPS.

9.1. Main contributions of the thesis

The main contributions of the thesis can be summarized as follows.

- In this thesis we formulated a theoretic framework for the modelling and analysis of distributed parameter systems. The framework is based on the port-Hamiltonian approach to systems theory and it is generalized from a mathematical analysis point of view. The key point in the approach is the structure of the model obtained, which is exploited in this thesis to provide a relative new point of view in the analysis of distributed parameter systems.

- The port-Hamiltonian formulation has been used to model different distributed parameter systems, showing that a large class of $1D$-systems can be studied using this approach. By using the port-Hamiltonian formulation one can group different systems according to the structure of the model, and in this way it is possible to provide some tools that are valid for a class of systems.

- The results presented in [LZM05] and [LZM04] have been extended to cover a larger class of systems, including flexible structures with (viscous or structural) damping and diffusion systems. This allows to parameterize passive boundary control systems in terms of matrices.

- We have shown that a large class of energy preserving systems are in turn conservative. This allows to relate stability, controllability, and observability properties of these systems.

- We provided tools that facilitate the verification of the Riesz basis property for a class of systems. The validity of this property leads to the verification of other properties of the system.

- It is shown that by making use of the structure of the model and the parametrization of the BCS described in Chapter 2, it is possible, in some cases, to simplify the verification of the stability property of some boundary control systems. In some cases, this verification can even be in terms of matrices. Furthermore, a simple result is provided to verify exponential stability.

- It has been described how to interconnect Dirac structures and how this helps in the study of interconnected systems. This property is useful when modeling systems using a modular approach where the system is thought of as the interconnection of smaller subsystems.

9.2. Recommendations for future work

The results presented here have led to more questions and therefore for several directions for future research. We briefly review them organized by topic.

9.2.1. Extension to nonlinear systems

This will clearly be a major research topic in the analysis of distributed parameters systems using the port-Hamiltonian approach. Proving existence of solutions for a class of nonlinear DPS would be already a big achievement.

As it was described in Chapter 1, the structure of the models are still valid for nonlinear systems. Hence one could try to exploit this (as it was done in this book) to try to prove similar results for nonlinear systems. Obviously, this is not an easy task. However, the author believes that it could be possible to get some interesting results in some cases. For instance, for the class of systems studied in Chapter 6, one could try to prove properties of a nonlinear port-Hamiltonian system by using the idea of the extended skew-symmetric operator, see Section 6.1. That is, by seeing the nonlinear system as the closed-loop of a linear system with a nonlinear feedback, the nonlinearity being included in the operator S.

Also, the fact that in some cases, the nonlinearity can be included in the operator $\mathcal{L}$ or S could be helpful and useful.

9.2.2. Some properties of distributed parameters systems

As it was shown in this thesis, some interesting results and tools that facilitate the verification of some properties of distributed parameter systems can be obtained using the port-Hamiltonian formulation. However, some of these results were general and others were restricted to the case $N = 1$. It would be desirable to extend the latter case to higher order differential operators, or at least to the case $N = 2$, which would include most of the cases already described in this book and the most common ones appearing in applications.

For instance, it is know that the Euler-Bernoulli beam under certain boundary conditions has the Riesz basis property. Thus, it should be possible to extend the results presented in Chapter 4 to the case $N = 2$.

Also, the author believes that the results on exponential stability presented at the end of Chapter 5, in particular Theorem 5.17 and Corollary 5.19, can be extended to the case of P_N not being invertible by using the Dirac structure obtained in Chapter 7. That is, by studying the system

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}\frac{\partial}{\partial z}\begin{bmatrix} \mathcal{L}_1 x_1 \\ \mathcal{L}_2 x_2 \end{bmatrix} + \begin{bmatrix} P_0 & \mathcal{G} \\ -\mathcal{G}^T & 0 \end{bmatrix}\begin{bmatrix} \mathcal{L}_1 x_1 \\ \mathcal{L}_2 x_2 \end{bmatrix},$$

where P_1 is a nonsingular $n \times n$ symmetric matrix and $\mathcal{G}$ is a $n \times m$ matrix, see (7.13), (7.14), and Example 7.8. These type of systems also appear in applications, such as models for laminated beams and the suspension system described in Example 7.8.

Even though, for energy preserving systems we have related stability with observability and controllability, it should also be possible to prove these properties directly. In particular, more tools for the analysis of the observability and controllability properties for systems including dissipation should also be provided. Also, it is needed to provide more results on stability for the class of systems in studied in Chapter 6. This should, in theory, be easier due to the dissipation phenomena that is present in these type of systems.

9.2.3. Interconnected systems

Clearly the presentation given in Chapter 7 was restricted to a specific class of power-conserving interconnection. Thus it would be desirable to understand better how more general types of interconnections fit into the framework presented in Chapter 7.

However, even in the case of the simple interconnection used in Chapter 7, there are some things that could be studied in more detail. For instance, the case of partial interconnection should not be too difficult. This corresponds to a plant interconnected with a controller where the control acts on part of the spatial domain, i.e., locally distributed feedback.

Other interested question regarding interconnected systems is whether it is possible to get an insight of the overall properties of the interconnected system by knowing only properties of the subsystems.

9.2.4. $2D$ and $3D$ systems

As discussed in Chapter 8, in this case there are some ideas that could be generalized. We know that these systems still have a similar structure to the one used in most of this book, so several ideas could be extended. However, a better understanding on how to select the input and output spaces is needed. It seems that if one selects either f_∂ or e_∂ as the boundary condition one can choose $H^{1/2}(\Gamma)^r$ or $H^{-1/2}(\Gamma)^r$, respectively, as the input space. However, when a linear combination of both is needed, then one may need to consider the space

$$\left\{ x \in H(J, \Omega) \mid \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in L_2(\Gamma)^{2r} \right\}$$

instead of just the space $H(J, \Omega)$. This leads to study further properties of this space. Also, the surjectivity of the boundary operator determining the input will follow if either f_∂ or e_∂ is surjective. However, proving existence of solutions is still not that simple, since the adjoint of the boundary operator is not easy to find in the general case.

Appendix A

Characteristic curves and Holmgren's Theorem

This appendix is mainly intended to give an idea of how Holmgren's Theorem can be used in the analysis of distributed parameter systems. Holmgren's Theorem is a useful tool in the study of stability and observability properties of some systems. In this appendix, only a brief introduction is presented just to get a flavor of how this theorem can be applied. For more details the reader is referred, for instance, to [Joh49], [Joh78], [Isa98] and [Hör93]. We start by describing how the characteristic curves affect the solution of a partial differential equation. Then, we introduce Holmgren's theorem in the case of constant coefficients.

A.1. Characteristic curves and PDEs

The characteristic curves are introduced and defined in Section 2.4. In this section we describe how they can be used to solve some PDEs. We do this by an example. Consider the equation

$$a(x,t)\frac{\partial v}{\partial x}(x,t) + b(x,t)\frac{\partial v}{\partial t}(x,t) = c(x,t)v(x,t) + d(x,t), \tag{A.1}$$

where a, b, c and d are given functions. Observe that the left hand side can be rewritten as

$$[a \quad b]\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial t} \end{bmatrix} = [a \quad b] \cdot \nabla v.$$

Hence, at each point (x, t) where the vector $[a,\ b]$ is defined and nonzero, the left side of (A.1) is a directional derivative of $v(x, t)$ in the direction of $[a,\ b]$. The equations

$$\frac{dx}{ds} = a(x, t), \qquad \frac{dt}{ds} = b(x, t) \tag{A.2}$$

determine a family of curves $x = x(s), t = t(s)$ whose tangent vector $[x'(s),\ t'(s)]$ coincides with the direction of the vector $[a,\ b]$ at each point where $[a,\ b]$ is defined and nonzero. Therefore, the derivative of $v(x, t)$ along this curves becomes

$$\frac{dv}{ds} = \frac{dv[x(s), t(s)]}{ds} = \frac{dv}{dx}\frac{dx}{ds} + \frac{dv}{dt}\frac{dt}{ds} = a\frac{dv}{dx} + b\frac{dv}{dt} = cv + d \tag{A.3}$$

where the chain rule was used as well as (A.1) and (A.2).

The method of characteristics for solving the initial value problem for (A.1) is as follows, see [Zau89, Ch.2 and 3] and [Col04, Ch. 5]. We assume that the initial curve C is given parametrically as

$$x = x(\tau), \quad t = t(\tau), \quad v = v(\tau) \tag{A.4}$$

for a given range of values of the parameter τ. The curve is required to have a continuous tangent vector at each point. Every value of τ fixes a point a point on C through which a unique characteristic curve passes. The family of characteristic curves determined by the points of C may be parametrized as

$$x = x(s, \tau), \quad t = t(s, \tau), \quad v = v(s, \tau), \tag{A.5}$$

with $s = 0$ corresponding to the initial curve C. That is, we have $x(0, \tau) = x(\tau)$, $t(0, \tau) = t(\tau)$, and $v(0, \tau) = v(\tau)$.

The equations (A.5), in general, yield a parametric representation of a surface in (x, t, v)-space that contains the initial curve C. Assuming the equations $x = x(s, \tau)$ and $t = t(s, \tau)$ can be inverted to give s and τ as (smooth) functions of x and t (this is the case if the Jacobian $\Delta(s, \tau) = \det\left[\begin{smallmatrix} x_s & x_\tau \\ t_s & t_\tau \end{smallmatrix}\right] \neq 0$ at C) these functions can be introduced into the equation $v(s, \tau)$. The resulting function $v = V(x, t)$ satisfies (A.1) in a neighborhood of the curve C in view of (A.3), the initial condition (A.5) (i.e., $V[x(\tau), t(\tau)] = v(\tau)$), and is the unique solution of the given initial value problem.

If the foregoing method does not lead to a solution, the initial value problem may not have a solution at all or it may have infinitely many solutions. The latter situation arises if the initial curve C is itself a characteristic curve.

Example A.1 Consider the equation

$$\frac{\partial v}{\partial t} = -c\frac{\partial v}{\partial x}, \quad v(x, 0) = f(x) \tag{A.6}$$

where c is a constant and $f(x)$ is a given function. To apply the method of characteristics we parametrize the initial curve C as follows, see (A.4),

$$x = \tau, \quad t = 0, \quad v = F(\tau). \tag{A.7}$$

The characteristic equations, i.e., (A.2) and (A.3), become

$$\frac{\partial x}{\partial s} = c, \quad \frac{\partial t}{\partial s} = 1, \quad \frac{\partial v}{\partial s} = 0. \tag{A.8}$$

Solving (A.8) subject to (A.7) with $s = 0$ corresponding to the initial curve, gives

$$x(s,\tau) = cs + \tau, \quad t(s,\tau) = s, \quad v(s,\tau) = F(\tau). \tag{A.9}$$

Using the first two equations to solve for s and τ as functions of x and t yields

$$s = t, \quad \tau = x - ct. \tag{A.10}$$

Substituting this in the equation for v in (A.9), we obtain

$$v(s,\tau) = F\left(\tau(x,t)\right) = F(x - ct). \tag{A.11}$$

Hence, we can see that the solution of (A.6) is given by $v(x,t) = F(x - ct)$. Using the initial condition $v(x,0) = f(x)$ we can see that the solution is

$$v(x,t) = f(x - ct), \ \forall t \geq 0 \text{ and } x \in (-\infty, \infty), \tag{A.12}$$

which is a function that moves to right with time. Also, observe that the solution is constant along the lines $x - ct = k$, where k is any constant, see Figure A.1. In general, the characteristics can be seen as a set of curves on which the solution remains constant.

Now consider $x \in [0,\infty)$ with the boundary condition $v(0,t) = h_1(t)$. We can see that the solution is still given by (A.11) as long as $x - ct > 0$, or equivalently $x > ct$. So, if $x_0 > ct_0$, we have $v(x_0,t_0) = F(x_0 - ct_0)$. Using the B.C. at $x = 0$, we obtain

$$\begin{aligned} v(0,t_0) &= F(-ct_0) = h_1(t_0) \quad \forall t_0 \geq 0 \\ \Rightarrow F(z) &= h_1(-\frac{1}{c}z) \quad \forall z \leq 0. \end{aligned} \tag{*}$$

Hence, $v(x_0,t_0) = h_1(-\frac{1}{c}x_0 + t_0)$ for all $x_0 - ct_0 \leq 0$. From this we can see that the boundary condition will move along the characteristic $-\frac{1}{c}x + t = k$, or equivalently $x - ct = -ck$, which is clearly parallel to $x - ct = k$. For instance, if $h_1(t) = 0$ and $f(x) = 1$, then we would see in Figure A.2 that the lighter region would have a value of 1 and the darker region would have a value of 0. Observe that $f(x)$ and $h_1(t)$ only meet at time $t = 0$. From this, it is easy to see that a B.C. at $x = b > 0$ will conflict with $F(\cdot)$ and hence with f and h_1, since this function and the B.C. would have to agree for all $t > 0$, see Figure A.2.

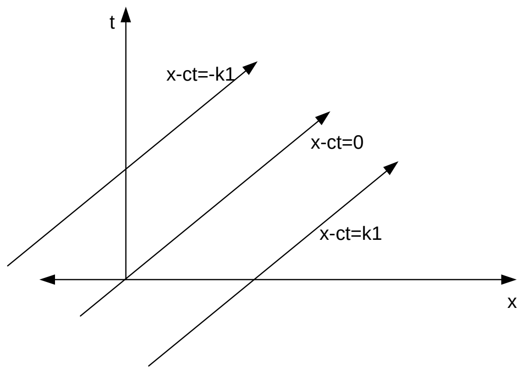

Figure A.1.: Characteristic curves.

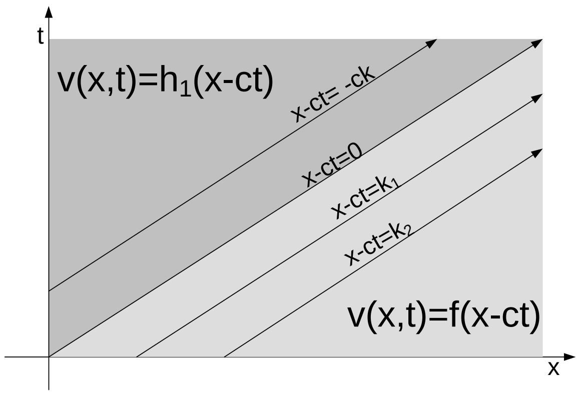

Figure A.2.: Characteristic curves.

A.2. Holmgren's Theorem (Constant Coefficients)

We follow the notation of Section 2.4. In particular, let

$$L(\mathbf{x}, D)u = \sum_{|\alpha| \leq m} A_\alpha(\mathbf{x}) D^\alpha u = B(\mathbf{x}). \tag{A.13}$$

The following theorem is an adaptation of the theorem on page 229 of [Joh49].

Theorem A.2: *Let L have constant coefficients. Let there be given a convex conical solid consisting of the convex hull of a convex 1-dimensional "area" $B = [0, T]$ in the plane $x_2 = 0$ and of a point $P = (x_1^0, x_2^0)$ with*

$$x_2^0 > 0, \quad x_1^0 \text{ in the interior of } B = [0, T].$$

If all hyperplanes through P that have no other point in common with the cone, are noncharacteristic, then the Cauchy data on B determines a solution of $Lu = 0$ uniquely at P.

In other words, the theorem above says that if a characteristic curve passing through a point P also intersects $B = [0, T]$, then the Cauchy data on B determines a solution of $Lu = 0$ uniquely at P, see Figures A.3 and A.4. The following

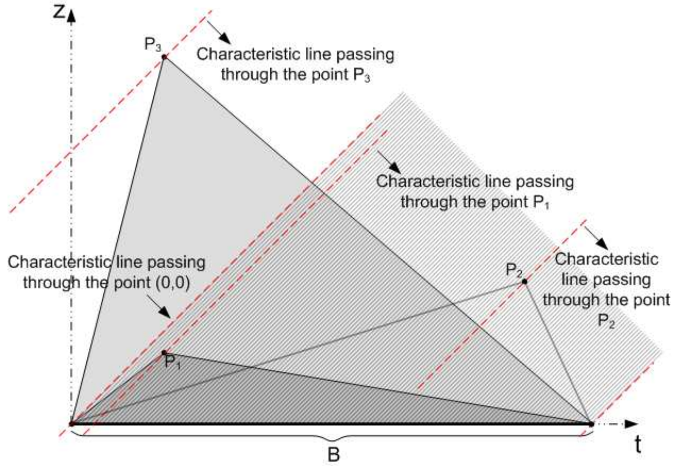

Figure A.3.: Characteristic lines passing through different points. Note that the characteristic lines passing through points in the shaded area intersect more than a point in the corresponding convex conical solid.

example shows how this theorem can be used.

Example A.3 (Transport equation revisited) Consider the differential equation

$$\frac{\partial v}{\partial t} = -\,c\frac{\partial v}{\partial z}, \quad v(0,z) = v_0(z), \quad z \in [0,b], \tag{A.14}$$
$$v(t,0) = 0, \qquad \forall\, t \geq 0.$$

First consider the case $c > 0$. Recall from Example A.1, that the characteristics in this case are given by the curves $z = ct + k$ with k a real constant. Following the theorem above and the boundary condition imposed on the PDE, we can see that in Figure A.3 v is zero along the line $(t, 0)$. Thus, we can deduce, from Holmgren's theorem that the solution v is zero in all points below the characteristic line passing through the point $(0, 0)$, see Figure A.3. Observe that the characteristic lines passing through the points above this line do not have any other point in common with respect to the corresponding convex conical solid.

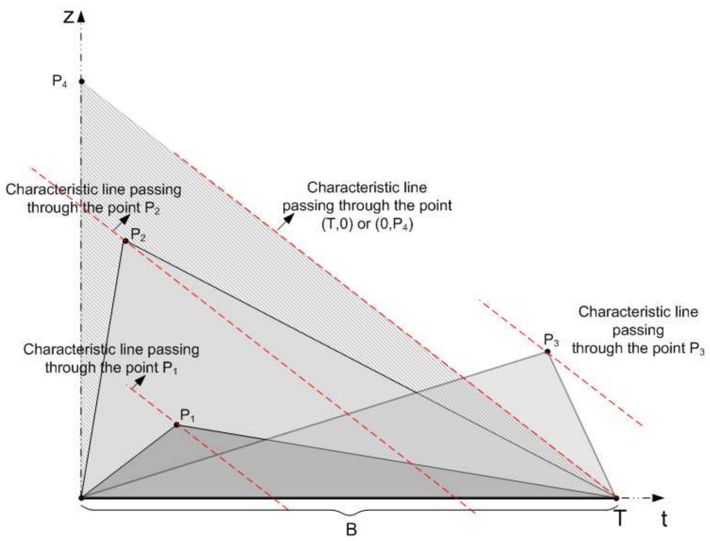

Figure A.4.: Characteristic lines passing through different points. Note that the characteristic lines passing through points below the line $(T, 0) - (0, P_4)$ intersect more than a point in the corresponding convex conical solid.

Next consider the case $c < 0$**.** In this case we have the equation $\frac{\partial v}{\partial t} = \tilde{c}\frac{\partial v}{\partial z}$ with the same boundary and initial condition as (A.14) and $\tilde{c} > 0$. Recall from Example A.1, that the characteristic curves in this case are given by the curves $z = -\tilde{c}t + k$ with k a real constant (the dotted lines in Figure A.4). Following the theorem above and the boundary condition imposed on the PDE, we can deduce that the solution v is zero in all points below the characteristic line passing through the point $(T, 0)$, see Figure A.4. *Since the boundary condition hold for all $t \geq 0$ (hence for any T), we can conclude, in particular, that $v_0(z)$ must be zero (for some T large enough), and hence the solution will be zero everywhere.* Observe that the characteristic lines passing through points above the line $(T, 0) - (0, P_4)$ do not have any other point in common with respect to the corresponding convex conical solid. $*$

A.2.1. Consequences of Holmgren's Theorem

Theorem A.4: *Consider the system*

$$\frac{\partial x}{\partial t}(t) = \mathcal{J} x(t), \qquad x(0) = x_0,$$

where the differential operator $\mathcal{J}$ is given by

$$\mathcal{J} x = \sum_{i=0}^{N} J_i \frac{d^i x}{dz^i}(z), \qquad z \in [a, b],$$

with J_N nonsingular. If this system satisfies the boundary conditions

$$x(a, t) = \frac{\partial x}{\partial z}(a, t) = \cdots = \frac{\partial^{N-1} x}{\partial z^{N-1}}(a, t) = 0, \tag{A.15}$$

$$x(b, t) = \frac{\partial x}{\partial z}(b, t) = \cdots = \frac{\partial^{N-1} x}{\partial z^{N-1}}(b, t) = 0, \quad \forall\, t \geq 0, \tag{A.16}$$

i.e., all boundary variables are zero, then $x_0 = 0$ and thus $x(t) = 0$ for all $t \geq 0$. $\diamondsuit$

PROOF: Recall that the only condition on J_N is that it is nonsingular. First we prove the case $N = 1$. In this case the principal symbol is $L_p\left(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right]\right) = \xi_1 I - \xi_2 J_1$. The characteristic lines are described by the lines $\phi = 0$ where $\phi = z + \lambda_i t + k$ with $k \in \mathbb{R}$ and λ_i, $i = 1, \ldots, n$, is an eigenvalue of J_1, see Section 2.4. Note that this are the same characteristics of Example A.3. That they are the characteristic curves follows since $\nabla \phi = \left[\begin{smallmatrix} \lambda_i \\ 1 \end{smallmatrix}\right]$ and $\det L_p\left(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \nabla \phi\right) = \det(\lambda_i I - J_1) = 0$, see Definition 2.22. One can apply Theorem A.2 to conclude that $x(t) = 0$. In fact, by using the same argument used in Example A.3 we can see that, in this case, we must have $x_0 = 0$. In other words, if $\lambda_i < 0$ then we can use the boundary conditions at b, see (A.15), to conclude that $x_i(z, 0) = 0$. Similarly, if $\lambda_j > 0$ then we can use the boundary conditions at a, see (A.16), to conclude that $x_j(z, 0) = 0$.

In the case $N > 1$, we have that the principal symbol is $L_p\left(\left[\begin{smallmatrix} t \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right]\right) = \xi_2^N J_N$. The characteristic lines are described by $\phi = t - c$. Since we have that x and its derivatives up to an order $N-1$ are zero on the lines $z=a$ and $z=b$ and the characteristic lines are perpendicular to those lines, we must have that x must be zero for all $t \geq 0$. ■

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Acknowledgment

I would like to express my deep and sincere gratitude to my supervisors, Professor A.J. van der Scahft and Dr. H. Zwart. Their wide knowledge, experience, and their logical way of thinking have been of great value for me. Their understanding, encouraging and personal guidance have provided a good basis for the present thesis. I am also deeply honored to have been supervised by both, as their immense knowledge and valuable contributions to the theory have always been a source of inspiration for me. I also appreciate the freedom they left me in doing my research during the past four years. I am very grateful for the careful reading and suggestions of the preliminary drafts of this thesis and for being so understanding when I decided to move to Oxford. Hans thank you for having the patience of introducing me to infinite-dimensional systems, for being always willing to solve my numerous questions, and for being so helpful now that I am in Oxford. Arjan thank you for introducing me to port-Hamiltonian systems and for helping me to understand better the physics behind it.

I would also like to thank the other members of my graduation committee, Prof. A. Bagchi , Prof. S. van Gils, Prof. R. Curtain, Prof. H. de Snoo, Prof. M. Tucsnak, and Prof. J. Winkin for agreeing to serve in the committee and reading the final draft of the thesis. In particular to Prof. Curtain for pointing out some mistakes and for the general suggestions about the thesis.

The research reported in this thesis was conducted under the auspices of the NWO through grant number 613.000.223. I would like to thank the NWO for funding me for the last four years.

During this work I have collaborated with many researchers for whom I have great regard, and I wish to extend my warmest thanks to all those who have helped me with my work. I am grateful to Dr. Anthony Holohan for introducing me into research and recommending me to follow my PhD in Twente University. I also wish to express my warm and sincere thanks to Yann Le Gorrec and Bernhard Maschke who made an important impact on my research during our continuous collaboration. Also for making possible my visits to Lyon University. I have some good memories of the times we meet and their hospitality during my visits to Lyon. Their continuous support helped me in all phases of my research for which I am infinitely indebted to them.

I would like to thank George Weiss for his collaboration and help during my visit to Imperial College. Also to the Control Training Site (CTS) for funding and making possible my visit to Imperial College in London. Also I am grateful to Waterloo University for funding part of my visit to the Workshop on Control of Infinite-Dimensional Systems held on Waterloo University, Canada. I also wish to thank Enrique Zuazua for his quick answers and help every time I needed it.

I am also indebted to the other staff members of the group in the University of Twente for sharing a friendly and pleasant research atmosphere with me. In particular, I would like to thank Jan Willem Polderman, Gjerrit Meinsma for helping me when I had technical or software related problems. I also want to thank the group secretary, Marja Langkamp, for taking care of the administrative matters and for the occasional chat.

I would also like to thank other people for making life as a PhD student more fun. First I would like to thank Ram, Vishy, Agung, Agoes, Ari, Emad, Hendra, Stefan, Maria, Simon, Norbert, Saikat, Jaroslav, Mikael, Eloisa and the other PhD students, ex-students, and visitors that have been involved with the group in the last four years for the friendly environment created at the group and all the memorable times. Ram it was nice to have you as my officemate and a good friend to me. I enjoyed our technical and non-technical discussions. Ari you have been a good friend and really helpful during my moving to Oxford. I really appreciate all your help.

Finally, I would like to thank those closest to me, whose support helped make the completion of this thesis possible. Thanks to my family, especially my mother, my brother, and Jairo who always were confident in me, and thanks to Amra for all the support that she gave me during the years I have been working on this thesis, especially for her love and patience during these years living apart. They all have giving me the best things in my life. I would like to dedicate this thesis to them.